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# Stochastic Hamiltonians for Non-Critical String Field Theories from Double-Scaled Matrix Models

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Abstract

We present detailed discussions on the stochastic Hamiltonians for non-critical string field theories on the basis of matrix models. Beginning from the simplest $c = 0$ case, we derive the explicit forms of the Hamiltonians for the higher critical case $k = 3$ (which corresponds to $c = -22/5$) and for the case $c = 1/2$, directly from the double-scaled matrix models. In particular, for the two-matrix case, we do not put any restrictions on the spin configurations of the string fields. The properties of the resulting infinite algebras of Schwinger-Dyson operators associated with the Hamiltonians and the derivation of the Virasoro and $W_3$ algebras therefrom are also investigated. Our results suggest certain universal structure of the stochastic Hamiltonians, which might be useful for an attempt towards a background independent string field theory.

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# 1. Introduction

A common idea towards a non-perturbative formulation of string theory is to start from the concept of string fields. Just as the ordinary local fields describe the motion and interaction of particles in terms of creation and annihilation operators, we can construct string field theories by appropriately slicing the world-sheets of strings and introducing the field operators to create and annihilate the strings. Clearly, there is continuously infinite amount of arbitrariness in choosing slicings. For instance, the light-cone string field theory [1] uses the light-like plane in the target spacetime to slice the world-sheet, while the covariant string field theories [2], in general, use different methods of slicing, which are based on the geometry of the moduli space of Riemann surfaces. The arbitrariness of slicing may be interpreted as a sort of gauge freedom of the theory. At present, however, we have no satisfactory framework to formulate such a gauge structure in a systematic and general way.

Recently, an interesting new way of slicing has been proposed [3], and the corresponding string field theories [4, 5] have been suggested for the case of noncritical strings with $c = 0$ and the case with minimal conformal matter $c = 1 - \frac{6}{m(m+1)}$. In this proposal, the world sheets are sliced by using a certain time parameter, which is intrinsically defined on the world sheet as a measure of the distances from the boundaries of the world sheets. In this paper, we will call the string field theories of this type as “proper time” string field theories (PSFTs), in analogy with the familiar proper-time representation of propagators in ordinary field theories. In PSFTs, it seems less difficult to incorporate the higher-genus (and hopefully, non-perturbative) effect than in the moduli space approach as employed in the ordinary covariant string field theories. Remarkably enough, there exists a single exact Hamilonian operator which directly characterizes all of the correlation functions in the system at once.

One of the many unsolved problems of present PSFTs, however, is that we do not know definite symmetry principles, if any, on the basis of which one can more or less uniquely characterize the theories. Thus, most of previous attempts had to rely upon certain guess works, and one can only justify the theories by checking the agreement of amplitudes with known results obtained from other methods, such as the matrix models. In this situation, the observation [6] that the PSFT for $c = 0$ can be interpreted as the collective field theory of matrix models formulated in stochastic quantization seems very useful and suggestive. In connection with this, we should recall an attractive idea [7] of relating the renormalization group formulation of string-field equations to stochastic quantization.

Another crucial question of PSFTs for further developing the theory is whether or not this method of slicing is meaningful for constructing string field theories for $c > 1$ and critical strings. The simplicity of proposed PSFTs for the case $c < 1$ is of course due to the simplicity of the target spaces. For example, in the case where the target space is the Ising model, one can deform the slicings such that the spin configuration on each string field is either all spin up or all spin down [8]. If one goes to $c > 1$, the slicing of this type would, however, be too singular to be tractable and one would have to introduce string fields without making any restrictions on possible matter configurations.

From this view point, it seems important to treat even the cases $c < 1$ without such restrictions and to study the structure of resulting PSFT, since we naturally expect that such a formulation should exhibit certain universal structure of the general PSFT which is common to PSFT for general critical strings. Since there is no known symmetry principle on the basis of which we can derive the theories, it is natural to directly derive such a formulation starting from the matrix models. That is what we shall present in this paper. Our hope is to get some insight on the nature of the PSFTs by deriving the formalism from the matrix models as explicitly as possible. We will follow the suggestion of ref. [6], using a slightly different approach, and construct the PSFT Hamiltonians directly by taking the double scaling limit of the matrix model Hamiltonians.*

In the next section, we will first review our method of deriving the stochastic Hamiltonian from the matrix models. We illustrate the method by using a simple quantum mechanical model with two degrees of freedom and point out some crucial assumptions required for proper-time string field theories. In section 3, we treat the case of one-matrix model and derive the Hamiltonians for the cases of $k = 2\,(c = 0)$ critical point and, as a simplest example of higher critical models, $k = 3$ critical point. In section 4, we discuss the Virasoro algebra structure associated with the Hamiltonians. Using the example with $k = 3$, we will clarify how the closed Virasoro algebra is obtained for higher critical cases. In section 5, we proceed to discuss the two-matrix model. Technically, this case is much more complicated than the case of one-matrix model and requires some new ingredients which have not shown up in the case of the one-matrix model. We will exhibit some interesting properties on the structure of the stochastic Hamiltonians, which may indeed be regarded as an example of the universal structure of the general PSFTs. In section 6, we will discuss the closure property of the infinite algebras associated with our Hamiltonians. Then, in section 7, the $W_3$ algebra of the two matrix model will be derived starting from the infinite algebra. These two sections provide consistency checks for the results of section 5, by deriving the expected properties of the two-matrix model from the present formalism. In the final section, we will conclude the paper by discussing possible implications of our work and remaining problems. Throughout this paper, we had to perform a number of tedious computations for which we could not find any appropriate references. Most of such details will be described in the Appendix.

## 2. The Hamiltonian of Stochastic Quantization

In this section, we will briefly introduce our method for deriving the Hamiltonian of PSFT. For clarity, we take a simple example of zero-dimensional field theory with two degrees of freedom $x, y$ with action $S(x, y)$,

$$Z = \int dxdy\, \mathrm{e}^{-S(x,y)}. \tag{2.1}$$

* For previous works which discuss the possibility of the PSFT with general matter configurations in the continuum formulation, see [5, 9].

The idea of stochastic quantization [10] can be summarized by introducing the following Hamiltonian,

$$H = -\frac{\partial}{\partial x}(\frac{\partial}{\partial x} + \frac{\partial S}{\partial x}) - \frac{\partial}{\partial y}(\frac{\partial}{\partial y} + \frac{\partial S}{\partial y}), \tag{2.2}$$

and the Fokker-Planck equation,

$$\frac{\partial}{\partial \tau}\Psi(x,y,\tau) = -H\Psi(x,y,\tau). \tag{2.3}$$

As is well known, the Fokker-Planck equation describes the statistical evolution of the probability distribution function $\Psi(x,y,\tau)$ for the system described by the stochastic equations of motion

$$\frac{dx}{d\tau} = -\frac{\partial S}{\partial x} + \eta_1, \tag{2.4}$$

$$\frac{dy}{d\tau} = -\frac{\partial S}{\partial y} + \eta_2, \tag{2.5}$$

where $\eta_1, \eta_2$ are Gaussian random noises. In the limit of $\tau \to \infty$, the solution of the Fokker-Planck equation reduces to the ground state

$$\Psi \to \mathrm{e}^{-S}, \tag{2.6}$$

satisfying $H\Psi = 0$ **under** the assumption that $\mathrm{e}^{S/2}\Psi$ rapidly decreases at infinity, corresponding to the positivity of the hermitian Laplace operator

$$\mathrm{e}^{S/2}H\mathrm{e}^{-S/2} = D_1^\dagger D_1 + D_2^\dagger D_2, \tag{2.7}$$

with $D_1 = \mathrm{e}^{-S/2}\frac{\partial}{\partial x}\mathrm{e}^{S/2}$, $D_2 = \mathrm{e}^{-S/2}\frac{\partial}{\partial y}\mathrm{e}^{S/2}$.

The Green function of an arbitrary observable $\mathcal{O}$ can be expressed as

$$\langle \mathcal{O} \rangle = \lim_{\tau\to\infty}\int dxdy\, \mathcal{O}(x,y)\Psi(x,y,\tau). \tag{2.8}$$

When (2.6) is assumed to be the unique ground state of the Hamiltonian $H$, we are entitled to suppose that the entire Schwinger-Dyson equation is replaced by a single ground-state condition given as

$$\lim_{\tau\to\infty}\int dxdy\, \mathcal{O}(x,y)H\Psi(x,y,\tau) = 0, \tag{2.9}$$

which is, after partial integrations, equivalent to

$$\int dxdy\, [\frac{\partial}{\partial x}\mathrm{e}^{-S}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\mathrm{e}^{-S}\frac{\partial}{\partial y}]\mathcal{O} = 0. \tag{2.10}$$

Using the generating functional with $\mathcal{O} = \mathrm{e}^{J_1 x + J_2 y}$, this is rewritten as

$$H(J, \frac{\partial}{\partial J})Z[J] = 0, \tag{2.11}$$

$$Z[J] = \int dxdy\, \mathrm{e}^{-S(x,y)+J_1x+J_2y}, \tag{2.12}$$

with

$$H(J, \frac{\partial}{\partial J}) = J_1T_1(J, \frac{\partial}{\partial J}) + J_2T_2(J, \frac{\partial}{\partial J}), \tag{2.13}$$

$$\begin{aligned} T_1 &= -S_x(\frac{\partial}{\partial J_1}, \frac{\partial}{\partial J_2}) + J_1, & (2.14)\\ T_2 &= -S_y(\frac{\partial}{\partial J_1}, \frac{\partial}{\partial J_2}) + J_2. & (2.15)\end{aligned}$$

From the assumption of the uniqueness of the solution for the ground state condition (2.11) (that we call Hamilton constraint), we can impose

$$T_1Z[J] = T_2Z[J] = 0. \tag{2.16}$$

which are nothing but the general form of the Schwinger-Dyson equation for our system,

$$\begin{aligned} T_1Z[J] &= \int dxdy\, \frac{\partial}{\partial x}\mathrm{e}^{-S(x,y)+J_1x+J_2y}, & (2.17)\\ T_2Z[J] &= \int dxdy\, \frac{\partial}{\partial y}\mathrm{e}^{-S(x,y)+J_1x+J_2y}. & (2.18)\end{aligned}$$

Note that the integrability condition is automatically satisfied,

$$[T_1, T_2] = S_{xy}(J, \frac{\partial}{\partial J}) - S_{yx}(J, \frac{\partial}{\partial J}) = 0. \tag{2.19}$$

In general, by introducing more general source terms $\sum_i J_i f_i(x, y)$, the stochastic Hamiltonian takes the following form with a general set of operators $T_i$

$$H = \sum_i J_i T_i. \tag{2.20}$$

Then, the assumption of uniqueness of the ground state implies that the partition function satisfies

$$T_iZ[J] = 0 \tag{2.21}$$

where

$$T_iZ[J] = \int dxdy\, (\frac{\partial}{\partial x}\frac{\partial f_i}{\partial x} + \frac{\partial}{\partial y}\frac{\partial f_i}{\partial y})\mathrm{e}^{-S+\sum_i J_if_i} \tag{2.22}$$

which can be, for appropriate choice of the source terms, expressed as functional differential operators in terms of $J_i$'s and are equivalent to the Schwinger-Dyson equations of the system. This should be regarded as a fundamental assumption of the method of stochastic quantization. We note that in general the algebra of the Schwinger-Dyson operators $T_i$ is non-Abelian.

Here, we add an important remark which partly underlies our later discussions. Namely, by introducing general source terms, together with this assumption, we can make the formalism background independent. After making a shift of the source function $J_i \to J_i + \delta_{i,S}$, the Hamiltonian equation (2.20)

$$HZ \equiv \int dxdy[\frac{\partial}{\partial x}\mathrm{e}^{-S}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\mathrm{e}^{-S}\frac{\partial}{\partial y}]\mathrm{e}^{S+\sum_i J_i f_i} = 0 \tag{2.23}$$

is then recast into the following form

$$T_S^0 Z + H^0 Z = 0, \tag{2.24}$$

where

$$H^0 Z \equiv \int dxdy(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\mathrm{e}^{\sum_i J_i f_i} = (\sum_i J_i T_i^0)Z. \tag{2.25}$$

Thus, under the assumption that the Hamilton equation is equivalent to the Schwinger-Dyson equations $T_i^0 Z = 0$, the first term of (2.24) vanishes and the Hamilton equation is reduced to $H^0 Z = 0$, a form which is formally independent of the starting action $S$. Here, $T_i^0$'s are the Schwinger-Dyson operators with the shifted source $J_i + \delta_{i,S}$, or in other words, with no bare action.

Now, to gain a more concrete understanding on the above assumption, let us consider a simple example with the bare action

$$S(x,y) = V(x) + V(y) + cxy, \quad V(x) = \frac{x^2}{2} + g\frac{x^3}{3}. \tag{2.26}$$

The operators $T_1, T_2$ are given as

$$T_1 = -\frac{\partial}{\partial J_1} - g\frac{\partial^2}{\partial J_1^2} - c\frac{\partial}{\partial J_2} + J_1, \tag{2.27}$$

$$T_2 = -\frac{\partial}{\partial J_2} - g\frac{\partial^2}{\partial J_2^2} - c\frac{\partial}{\partial J_1} + J_2. \tag{2.28}$$

On the other hand, from the view point of the Schwinger-Dyson equations, it is easy to check that the following set of equations gives a closed recursion equation for $\langle x^n \rangle$:

$$0 = \int dxdy\frac{\partial}{\partial x}x^n\mathrm{e}^{-S}, \tag{2.29}$$

$$0 = \int dxdy\frac{\partial}{\partial y}x^n\mathrm{e}^{-S}, \tag{2.30}$$

$$0 = \int dxdy\frac{\partial}{\partial x}x^n y\mathrm{e}^{-S}, \tag{2.31}$$

which are obtained from the $T_1, T_2$ constraints by making a power series expansion in $J_1, J_2$ as

$$\frac{\partial^n}{\partial J_1^n}T_1 Z[J]\Big|_{J=0} = 0, \tag{2.32}$$

$$\frac{\partial^n}{\partial J_1^n} T_2 Z[J]\Big|_{J=0} = 0, \tag{2.33}$$
$$\frac{\partial^{n+1}}{\partial J_1^n \partial J_2} T_1 Z[J]\Big|_{J=0} = 0, \tag{2.34}$$

respectively. The closed recursion equation for $\langle x^n \rangle$ is obtained by expressing the correlators of the form $\langle x^n y \rangle, \langle x^n y^2 \rangle$ using the last two equations in terms of $\langle x^n \rangle$ and by substituting the results into the first equation.†

However, it is not difficult to see that we cannot derive all of these conditions (2.29)∼(2.31) directly by taking finite order derivatives with respect to $J_1, J_2$ from the single Hamiltonian constraint,

$$(J_1 T_1 + J_2 T_2) Z[J] = 0. \tag{2.35}$$

For example, by taking a derivative $\frac{\partial^{n+1}}{\partial J_1^n \partial J_2}$ of (2.35), we obtain the sum of (2.33) and (2.34), but can never obtain (2.33) or (2.34), separately, by taking any derivatives of finite order.

Thus, the equivalence of the Hamiltonian constraint (2.35) with the Schwinger-Dyson equations (2.16) is based on the assumption of the uniqueness of the solution for (2.35) which requires that $\mathrm{e}^{-S/2}$ rapidly decreases at infinity $x, y \to \pm\infty$. This uniqueness assumption amounts to setting certain conditions on the partition function $Z[J]$ which **cannot** be expressed in any **finite** order of the expansion with respect to the source functions $J_i$. If the appropriate global conditions for the uniqueness are not satisfied, the stochastic Hamiltonian would fail‡ to give a unique ground state in the limit $\tau \to \infty$, and the limit would, in general, depend on the choice of the initial state.

In the case of simple quantum mechanical models, it is relatively easy to identify the necessary global conditions. However, in more complex systems such as the double-scaling limit of matrix models, it is quite nontrivial to state such conditions, and, in fact, there has been no known result replying this question.

It is clear that the PSFT proposed in ref. [4] is based on the tacit assumptions of similar nature as above. Unless the Hamiltonian is able to define a more or less unique ground state under the same constraint as for the Schwinger-Dyson equations, the concept of the PSFT Hamiltonians would become less significant, since in that case we have to recourse to the Schwinger-Dyson equations themselves for the definition of the theory.

In the following sections, we will discuss the Hamiltonians of PSFT for one- and two-matrix models using the above methods, keeping those assumptions in mind. We here mention that our method can be translated into the language of ref. [4] by making a functional transformation

$$Z[J] \to \langle Z| = \langle 0| \exp(\sum \psi_i \frac{\partial}{\partial J_i}) Z[J]\Big|_{J=0}. \tag{2.36}$$

† This procedure is essentially the same as the one employed in ref. [20] to derive a closed subset of the Schwinger-Dyson equations for the two-matrix model.

‡ This is obvious for the case of usual laplacian $\triangle = \sum_i \partial_i \partial_i$. There are an infinite number of polynomial solutions for $\triangle f = 0$. The uniqueness can be guaranteed under the requirement, say, of $L^2$ normalizability condition.

The source functions and their derivatives are replaced by the annihilation- and creation-string fields

$$J_i \leftrightarrow \psi_i,\ \frac{\partial}{\partial J_i} \leftrightarrow \psi_i^\dagger, \tag{2.37}$$

respectively. Then, the correlators are given as

$$\langle Z|\psi_1^\dagger\psi_2^\dagger\cdots\psi_n^\dagger|0\rangle = \Big(\prod_{i=1}^n \frac{\partial}{\partial J_i}\Big)Z[J]\Big|_{J=0}. \tag{2.38}$$

Note that under the above transformation the normal ordering $J\cdots\frac{\partial}{\partial J}\cdots$ for $J, \frac{\partial}{\partial J}$ is automatically transformed into the one $\psi^\dagger\cdots\psi\cdots$ for $\psi, \psi^\dagger$. The Hamilton constraint is thus $\langle Z| : H(\psi,\psi^\dagger) := 0$, and the state $\langle Z|$ is obtained as $\langle Z| = \lim_{\tau\to\infty}\langle 0|\mathrm{e}^{-\tau :H:}$.

# 3. PSFTs from One-Matrix Model

In this section, we derive the stochastic Hamiltonians from the one-matrix model at the $k = 2$ and $k = 3$ critical points which correspond to the matter central charges $c = 0$ and $c = -22/5$, respectively.

## *3.1 Stochastic Hamiltonian at $c = 0$*

We first treat the case of $c = 0$. Although this case has already been discussed in ref. [6] within the framework of the collective field method, we present some details for the purpose of explaining our method which is slightly different from ref. [6].

The generating functional of the $c = 0$ one-matrix model is defined by

$$\begin{aligned} Z[J] &= \frac{1}{Z}\int d^{N^2}M\ \mathrm{e}^{-N\mathrm{tr}V(M)}\mathrm{e}^{J\cdot\Phi}, \\ Z &= \int d^{N^2}M\ \mathrm{e}^{-N\mathrm{tr}V(M)}, \end{aligned} \tag{3.1}$$

$$V(M) = \frac{1}{2}M^2 - \frac{g}{3}M^3, \tag{3.2}$$

$$J\cdot\Phi = \int_L \frac{d\zeta}{2\pi i}J(\zeta)\Phi(\zeta), \tag{3.3}$$

where

$$\Phi(\zeta) = \frac{1}{N}\mathrm{tr}\frac{1}{\zeta - M} \tag{3.4}$$

is a loop operator and the contour of $\zeta$-integral $L$ is chosen to be parallel to the imaginary axis such that in the region of the right of $L$ there are no poles of $\Phi(\zeta)$. The source function

$J(\zeta)$ can take an arbitrary form as a function on $L$. The variable $\zeta$ can be regarded as being conjugate to the length of the loop in the sense of Laplace transform.

We start with the ground state condition of the stochastic Hamiltonian

$$0 = -\frac{1}{Z}\int d^{N^2}M \sum_{\alpha=1}^{N^2} \frac{\partial}{\partial M_\alpha}\left(\mathrm{e}^{-N\mathrm{tr}V(M)}\frac{\partial}{\partial M_\alpha}\mathrm{e}^{J\cdot\Phi}\right) \tag{3.5}$$

where $M$ is expanded by the basis of $N\times N$ hermitian matrices $\{t^\alpha\}$:

$$M = \sum_{\alpha=1}^{N^2} M_\alpha\, t^\alpha.$$

Using the identities

$$\begin{aligned}\sum_\alpha \mathrm{tr}(At^\alpha Bt^\alpha) &= \mathrm{tr}A\,\mathrm{tr}B,\\ \sum_\alpha \mathrm{tr}(At^\alpha)\mathrm{tr}(Bt^\alpha) &= \mathrm{tr}AB,\end{aligned}$$

we obtain formulas such as

$$\sum_\alpha \frac{\partial}{\partial M_\alpha}\Phi(\zeta)\frac{\partial}{\partial M_\alpha}\Phi(\zeta') = -\frac{1}{N}\partial_\zeta\partial_{\zeta'}D_z(\zeta,\zeta')\Phi(z), \tag{3.6}$$

$$\sum_\alpha \frac{\partial^2}{\partial M_\alpha^2}\Phi(\zeta) = -N\frac{\partial}{\partial\zeta}\Phi(\zeta)^2, \tag{3.7}$$

$$\frac{1}{N}\sum_\alpha \mathrm{tr}(M^n t^\alpha)\mathrm{tr}(t^\alpha\frac{1}{(\zeta-M)^2}) = -\frac{d}{d\zeta}(\zeta^n\Phi(\zeta) - \sum_{k=0}^{n-1}\zeta^k\frac{1}{N}\mathrm{tr}(M^{n-1-k})), \tag{3.8}$$

where the symbol $D_z$ is the so-called combinatorial derivative, defined as

$$D_z(\zeta,\zeta')f(z) \equiv \frac{f(\zeta)-f(\zeta')}{\zeta-\zeta'}, \tag{3.9}$$

which appears when two loops merge into a new loop.

We can then reduce eq. (3.5) to a functional differential equation with respect to the source $J(\zeta)$,

$$0 = \mathcal{H}Z[J] \tag{3.10}$$

$$\begin{aligned}\mathcal{H} = {} & \int_L \frac{d\zeta}{2\pi i}J(\zeta)\partial_\zeta\left[\left(\frac{\delta}{\delta J(\zeta)} - \frac{1}{2}(\zeta-g\zeta^2)\right)^2 - \frac{1}{4}(\zeta-g\zeta^2)^2 - g\zeta\right]\\ & + \frac{1}{N^2}\int_L\frac{d\zeta}{2\pi i}J(\zeta)\int_L\frac{d\zeta'}{2\pi i}J(\zeta')\partial_\zeta\partial_{\zeta'}D_z(\zeta,\zeta')\frac{\delta}{\delta J(z)}.\end{aligned} \tag{3.11}$$

The functional derivative $\frac{\delta}{\delta J(\zeta)}$ is defined for $\zeta$ on the contour and acts on the source $J(\zeta')$ as a delta function

$$\frac{\delta J(\zeta')}{\delta J(\zeta)} = 2\pi i \delta(\zeta - \zeta'). \tag{3.12}$$

when both $\zeta$ and $\zeta'$ resides on the same contour. $\mathcal{H}$ is the exact stochastic Hamiltonian for the $c = 0$ PSFT before taking the double-scaling limit. The first term represents the splitting process of a loop, while the second represents the merging process of two loops. Note that the expression (3.11) is normal ordered in the sense that the differential operators $\frac{\delta}{\delta J(\zeta)}$ always sit right of $J(\zeta)$.

Introducing a lattice spacing $a$, we now take the continuum limit (the double scaling limit) $a \to 0$ by defining the scaling variables as

$$\zeta = \zeta_*(1 + ay), \quad g = g_*(1 - a^2 t), \quad \frac{1}{N} = a^{5/2} g_{\rm st}$$

where the critical points are

$$\zeta_* = (\sqrt{3} + 1) \cdot 3^{1/4}, \quad g_* = \frac{3^{1/4}}{6}.$$

The meanings of the variables $y$, $t$ and $g_{\rm st}$ are the Laplace conjugate of loop length, the cosmological constant, and the string coupling constant, respectively. From the result of the disk amplitude [11], it can be seen that the contribution of the poles of $\Phi(\zeta)$ accumulates to a cut of the interval $[-(\sqrt{3} - 1) \cdot 3^{3/4} + O(a^2),\ \zeta_* - 4 \cdot 3^{-1/4} a \sqrt{t} + O(a^2)]$. The region $\mathrm{Re}\,\zeta \geq \zeta_*$ contains no singularities of $\Phi(\zeta)$. So, in the scaling limit we can choose as the contour $L$ the line $[\zeta_* - i\infty, \zeta_* + i\infty]$, which is mapped to the imaginary axis in $y$-plane.

In order to obtain the correct continuum limit, we have to subtract a non-universal part from the correlation functions. In the present case, it is required only for the one-point disk amplitude. Namely, the connected $K$-point function

$$W(\zeta_1, \cdots, \zeta_K) = \langle \Phi(\zeta_1) \cdots \Phi(\zeta_K) \rangle_c$$

is written as

$$\begin{aligned} W(\zeta) &= \frac{1}{2}(\zeta - g\zeta^2) + a^{3/2} w(y) + O(a^2) \\ W(\zeta_1, \cdots, \zeta_K) &= a^{3K/2} w(y_1, \cdots, y_K) + O(a^{(3K+1)/2}) \quad (K \geq 2) \end{aligned}$$

where $\frac{1}{2}(\zeta - g\zeta^2)$ is the non-universal part of the disk amplitude, and $w$ is the universal part giving the correct continuum limit.

Thus we redefine the source $\tilde{J}(y)$ and the functional derivative by

$$\frac{\delta}{\delta J(\zeta)} = \frac{1}{2}(\zeta - g\zeta^2) + a^{3/2} \frac{\delta}{\delta \tilde{J}(y)}, \tag{3.13}$$

$$J(\zeta) = \zeta_*^{-1} a^{-5/2} \tilde{J}(y). \tag{3.14}$$

The shift (3.13) corresponds to the rescaling of the partition function as $Z[J] = \exp(\int_L \frac{d\zeta}{2\pi i} J(\zeta)(\zeta - g\zeta^2)/2) Z[\tilde{J}(y)]$. Then, $\mathcal{H}$ becomes

$$\begin{aligned}\mathcal{H} &= a^{1/2}\zeta_*^{-1}\Bigg[\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}(y)\partial_y\left(\frac{\delta^2}{\delta\tilde{J}(y)^2} - C(y^3 - \frac{3}{4}Ty) + O(a^1)\right)\\ &\quad + g_{\rm st}^2\zeta_*^{-2}\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}(y)\int_{-i\infty}^{i\infty}\frac{dy'}{2\pi i}\tilde{J}(y')\partial_y\partial_{y'}D_z(y,y')\frac{\delta}{\delta\tilde{J}(z)}\Bigg],\end{aligned}$$

where

$$T = \frac{16}{3(1+\sqrt{3})^2}t, \quad C = \frac{\sqrt{3}}{12}(1+\sqrt{3})^3.$$

Note that in the merging interaction (namely, the term of the form $JJ\frac{\delta}{\delta J}$) the shift of the functional derivative does not contribute because

$$\partial_\zeta\partial_{\zeta'}D_z(\zeta,\zeta')\frac{1}{2}(z - gz^2) = 0. \tag{3.15}$$

After finite rescalings

$$\tilde{J}(y) \to \tilde{J}(y)C^{-1/2}, \quad \frac{\delta}{\delta\tilde{J}(y)} \to \frac{\delta}{\delta\tilde{J}(y)}C^{1/2}, \quad g_{\rm st} \to g_{\rm st}\zeta_* C^{1/2},$$

we have the stochastic Hamiltonian in the continuum theory

$$\begin{aligned}\mathcal{H} &= \int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}(y)\partial_y\frac{\delta^2}{\delta\tilde{J}(y)^2} - \int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}(y)\tilde{\rho}(y)\\ &\quad + g_{\rm st}^2\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}(y)\int_{-i\infty}^{i\infty}\frac{dy'}{2\pi i}\tilde{J}(y')\partial_y\partial_{y'}D_z(y,y')\frac{\delta}{\delta\tilde{J}(z)}, \qquad (3.16)\\ \tilde{\rho}(y) &= 3y^2 - \frac{3}{4}T, \qquad (3.17)\end{aligned}$$

where the overall factor $a^{1/2}\zeta_*^{-1}C^{1/2}$ was absorbed by a redefinition of the fictitious time. This result, which has been already known from ref. [6], essentially coincides with the form of the $c = 0$ non-critical string field theory[§] proposed by Ishibashi and Kawai [4], if one uses the Laplace-transformed string fields instead of their loop-length representation.

### *3.2 PSFT for a Higher Critical One-Matrix Model*

We next treat a case of higher critical point ($k = 3(c = -22/5)$).[¶] This problem is interesting since a naive extension of the $c = 0$ hamiltonian leads to an apparent contradiction as discussed in ref. [15].

[§] For a derivation of the $c = 0$ Hamiltonian directly from dynamical triangulation, see ref. [12].

[¶] Extension of the formalism of ref. [3] to higher critical cases has been given in [13].

The $k=3$ critical theory is realized by the potential of the fourth-degree polynomial

$$V(M)=\frac{\beta}{N}(\frac{g_2}{2}M^2+\frac{g_3}{3}M^3+\frac{1}{20}M^4), \tag{3.18}$$

where $g_3$ is a real solution of the cubic equation

$$25g_3^3-30g_3+32=0,$$

or explicitly

$$g_3=-\left(\frac{2}{25}\right)^{1/3}((8+3\sqrt{6})^{1/3}+(8-3\sqrt{6})^{1/3}), \tag{3.19}$$

and

$$g_2=\frac{5g_3^2-2}{3}. \tag{3.20}$$

Note that we do not use the well known even critical potential (of sixth order) at $k=3$ critical point, in order to avoid a complication caused by the $Z_2$ symmetry.[‖]

Now, let us derive the Hamiltonian for the $k=3$ critical case. Considering the generating functional (3.1) with the potential (3.18), we have the Hamiltonian before taking the scaling limit:

$$\begin{aligned}\mathcal{H} \;=\; & \int_L \frac{d\zeta}{2\pi i}J(\zeta)\partial_\zeta\left[\left(\frac{\delta}{\delta J(\zeta)}-\frac{1}{2}V'(\zeta)\right)^2-\frac{1}{4}V'(\zeta)^2\right.\\ & \left.+\frac{\beta}{N}(g_3\zeta+\frac{1}{5}\zeta^2)+\frac{\beta}{N}\frac{1}{5}\zeta\oint\frac{d\zeta'}{2\pi i}\zeta'\frac{\delta}{\delta J(\zeta')}\right]\\ & +\frac{1}{N^2}\int_L\frac{d\zeta}{2\pi i}J(\zeta)\int_L\frac{d\zeta'}{2\pi i}J(\zeta')\partial_\zeta\partial_{\zeta'}D_z(\zeta,\zeta')\frac{\delta}{\delta J(z)},\end{aligned} \tag{3.21}$$

where $\oint$ is the integral over the contour encircling the poles of $\Phi(\zeta)$. Note that $\oint\frac{d\zeta'}{2\pi i}\zeta'\frac{\delta}{\delta J(\zeta')}$ corresponds to the insertion of a microscopic loop represented by the operator $\frac{1}{N}\mathrm{tr}M$. In the $c=0$ case, no such term appears if one uses the third order potential of (3.2), because of the formula (3.8).

Next, we need to identify the non-universal parts of loop operators in the scaling limit $a\to 0$ defined by

$$\frac{N}{\beta}=1-a^3t, \tag{3.22}$$

$$\zeta=\zeta_*(1+ay), \quad \zeta_*=\frac{-5g_3+2}{3}. \tag{3.23}$$

‖ We do not know any previous work discussing the $k=3$ critical point using the quartic potential. For a brief explanation of the derivation of the quartic critical potential, see the Appendix A.

From the results of the Appendix A, we have

$$\langle \Phi(\zeta) \rangle_0 = \frac{1}{2}V'(\zeta) + a^{5/2}w(y) + O(a^{7/2}), \tag{3.24}$$

$$\left\langle \frac{1}{N}\mathrm{tr}M \right\rangle_0 = -\frac{32+25g_3}{15} - a^3\frac{4}{5}t + a^4\frac{3}{4}t^{4/3} + O(a^5), \tag{3.25}$$

where, in the disk amplitude corresponding to a macroscopic loop, the first term $\frac{1}{2}V'(\zeta)$ is the non-universal part, while the second term $w(y)$ denotes the universal one:

$$w(y) = -\frac{1}{5}\zeta_*^{5/2}(y^2 - \frac{1}{2}T^{1/3}y + \frac{3}{8}T^{2/3})\sqrt{y+T^{1/3}} \tag{3.26}$$

with $T = (2\zeta_*^{-1})^3 t$. For the microscopic disk amplitude (3.25), the first two terms represents the non-universal part, and the third term is universal.

Using the above results, we see that the source $\tilde{J}(y)$ and the microscopic loop operator $\mathcal{O}_0$ in the continuum theory should be defined by

$$\frac{\delta}{\delta J(\zeta)} = \frac{1}{2}V'(\zeta) + a^{5/2}\frac{\delta}{\delta \tilde{J}(y)}, \quad J(\zeta) = a^{-7/2}\zeta_*^{-1}\tilde{J}(y), \tag{3.27}$$

$$\oint \frac{d\zeta'}{2\pi i}\zeta'\frac{\delta}{\delta J(\zeta')} = -\frac{32+25g_3}{15} - a^3\frac{4}{5}t + a^4\mathcal{O}_0. \tag{3.28}$$

The string coupling constant is introduced as

$$\frac{1}{N} = a^{7/2}g_{\mathrm{st}}. \tag{3.29}$$

Substituting (3.22), (3.23) and these rescaled expressions into the lattice Hamiltonian (3.21), we obtain

$$\begin{aligned} \mathcal{H} = {} & a^{3/2}\zeta_*^{-1}\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}(y)\partial_y\left[\frac{\delta^2}{\delta\tilde{J}(y)^2} - \frac{1}{25}\zeta_*^5(y^5 + \frac{5}{8}Ty^2) + \frac{1}{5}\zeta_* y\mathcal{O}_0\right] \\ & + a^{3/2}\zeta_*^{-3}g_{\mathrm{st}}^2\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}(y)\int_{-i\infty}^{i\infty}\frac{dy'}{2\pi i}\tilde{J}(y')\partial_y\partial_{y'}D_z(y,y')\frac{\delta}{\delta\tilde{J}(z)} \\ & + O(a^2), \end{aligned} \tag{3.30}$$

where as in the $c=0$ case we chose the line $[\zeta_* - i\infty,\ \zeta_* + i\infty]$ as the contour $L$. Note that in the merging term the shift of the derivative $\frac{\delta}{\delta J}$ produces a quadratic term with respect to $\tilde{J}$, but it does not contribute to the leading term of $\mathcal{H}$ as $a \to 0$, because

$$\begin{aligned} & \frac{1}{N}\int_L \frac{d\zeta}{2\pi i}J(\zeta)\int_L\frac{d\zeta'}{2\pi i}J(\zeta')\partial_\zeta\partial_{\zeta'}D_z(\zeta,\zeta')\frac{1}{2}V'(\zeta) \\ & = a^2 g_{\mathrm{st}}^2\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}(y)\int_{-i\infty}^{i\infty}\frac{dy'}{2\pi i}\tilde{J}(y')\frac{1}{10}. \end{aligned} \tag{3.31}$$

After making the rescalings again,

$$\tilde{J}(y) \to (\frac{1}{5}\zeta_*^{5/2})^{-1}\tilde{J}(y), \quad \frac{\delta}{\delta\tilde{J}(y)} \to \frac{1}{5}\zeta_*^{5/2}\frac{\delta}{\delta\tilde{J}(y)},$$

$$\mathcal{O}_0 \to -\frac{1}{5}\zeta_*^4\mathcal{O}_0, \quad g_{\rm st} \to \frac{1}{5}\zeta_*^{7/2}g_{\rm st}, \tag{3.32}$$

we finally obtain the continuum Hamiltonian

$$\begin{aligned}\mathcal{H} &= \int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}(y)\partial_y\frac{\delta^2}{\delta\tilde{J}(y)^2} - \int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}(y)\tilde{\rho}(y) \\ &\quad + g_{\rm st}^2\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}(y)\int_{-i\infty}^{i\infty}\frac{dy'}{2\pi i}\tilde{J}(y')\partial_y\partial_{y'}D_z(y,y')\frac{\delta}{\delta\tilde{J}(z)}, \qquad (3.33)\\ \tilde{\rho}(y) &= 5y^4 + \frac{5}{4}Ty + \mathcal{O}_0, \qquad (3.34)\end{aligned}$$

where the overall factor $a^{3/2}\frac{1}{5}\zeta_*^{3/2}$ was absorbed into the fictitious time.

We emphasize that in this Hamiltonian, the tadpole term $-\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}(y)\tilde{\rho}(y)$, is not a pure c-number, but contains the $y$-independent operator $\mathcal{O}_0$. This is in contrast to the $c=0$ case where the tadpole term consists only of the c-number function. Actually, the "operator" part of the tadpole is a misnomer. We should rather call it a kinetic term. The Hamiltonian description of the higher critical point requires a kinetic term for infinitesimally small loop, in addition to the genuine tadpole corresponding to the c-number part of $\tilde{\rho}$.

In the Appendix B, we will determine the operator $\mathcal{O}_0$ using the Schwinger-Dyson equations. And in the next section, using this result, we confirm that it is just necessary for ensuring the closure of the algebra of the Schwinger-Dyson operators appearing in the Hamiltonian. As is discussed in ref. [15] in trying the extension of the $c=0$ Hamiltonian to the higher critical case, the integrability condition would not be satisfied if one had naively replaced the $\tilde{\rho}(y)$ of the $c=0$ case with c-number polynomials of higher degree. The authors in ref. [15] proposed a possible way out, which is, however, different from ours.

Before concluding this section, we derive the disk amplitude from the PSFT Hamiltonian (3.33) and compare with the matrix model result as a consistency check of our result. In the sphere approximation, the condition

$$\left.\frac{\delta}{\delta\tilde{J}(y)}\mathcal{H}Z[J]\right|_{J=0} = 0, \tag{3.35}$$

is reduced to the equation for the one-point function $w(y)$,

$$\partial_y[w(y)^2 - (y^5 + \frac{5}{8}Ty^2 + y\langle\mathcal{O}_0\rangle_0)] = 0. \tag{3.36}$$

By demanding that the cut of $w(y)$ resides only on the real axis, as is required from the original definition of the loop operator (3.4), both of $w(y)$ and the expectation value of $\mathcal{O}_0$

are uniquely determined as

$$w(y) = (y^2 - \frac{1}{2}T^{1/3}y + \frac{3}{8}T^{2/3})\sqrt{y+T^{1/3}}, \tag{3.37}$$

$$\langle \mathcal{O}_0 \rangle_0 = -\frac{15}{64}T^{4/3}, \tag{3.38}$$

which coincide with the results (3.26) and (3.25) obtained directly without using the Hamiltonian, after taking account the rescalings (3.32).

## 4. Derivation of the Virasoro Constraints

In this section, we examine the integrability condition of the Schwinger-Dyson operators associated with the Hamiltonian equation,

$$\mathcal{H}Z[J] = 0 \tag{4.1}$$

for $k = 2, 3$ cases. As a warmup exercise, let us begin from the simplest case of $c = 0$.

### *4.1 The Virasoro algebra at $k = 2$*

Recalling the discussions of section 2, we rewrite the Hamiltonian $\mathcal{H}$ in the form

$$\mathcal{H} = -\int_{-i\infty}^{i\infty} \frac{dy}{2\pi i} \tilde{J}(y)\partial_y T(y), \tag{4.2}$$

$$\partial_y T(y) = \partial_y T_0(y) + \tilde{\rho}(y), \tag{4.3}$$

$$T_0(y) = -\frac{\delta^2}{\delta \tilde{J}(y)^2} - g_{\rm st}^2 \int_{-i\infty}^{i\infty} \frac{dy'}{2\pi i} \tilde{J}(y')\partial_{y'} D_z(y, y')\frac{\delta}{\delta \tilde{J}(z)}. \tag{4.4}$$

Thus the Schwinger-Dyson equation associated with the $c = 0$ Hamiltonian is

$$\partial_y T(y) Z[J] = 0. \tag{4.5}$$

After a straightforward calculation using the functional derivative

$$\frac{\delta \tilde{J}(y)}{\delta \tilde{J}(y')} = 2\pi i \delta(y - y'), \tag{4.6}$$

we find a closed algebra for $T_0$ operators

$$[\partial_{y_1} T_0(y_1), \partial_{y_2} T_0(y_2)] = -g_{\rm st}^2 \partial_{y_1}\partial_{y_2}(\partial_{y_1} - \partial_{y_2})\frac{1}{y_1 - y_2}(T_0(y_1) - T_0(y_2)), \tag{4.7}$$

The algebra of $\partial_y T(y)$ is obtained by substituting

$$T_0(y) = T(y) - \int^y dy' \tilde{\rho}(y')$$

into (4.7). Then, using the explicit expression of $\tilde{\rho}$ (3.17) we find that the effect of $\tilde{\rho}$ vanishes

$$\partial_{y_1}\partial_{y_2}(\partial_{y_1} - \partial_{y_2}) D_z(y_1, y_2) \int^z dy \tilde{\rho}(y) = 0. \tag{4.8}$$

Thus, $\partial_y T(y)$ forms the same closed algebra as $\partial_y T_0(y)$

$$[\partial_{y_1} T(y_1), \partial_{y_2} T(y_2)] = -g_{\rm st}^2 \partial_{y_1}\partial_{y_2}(\partial_{y_1} - \partial_{y_2}) \frac{1}{y_1 - y_2} (T(y_1) - T(y_2)). \tag{4.9}$$

This agrees with the Laplace-transformed version of the result in ref. [8].

### *4.2 The Schwinger-Dyson operators at $k = 3$*

The $k = 3$ case is less trivial. The only difference from the $k = 2$ ($c = 0$) case lies in $\tilde{\rho}$ (3.34). Namely, the c-number part of the tadpole term $5y^4$ in $\tilde{\rho}$ gives a non-vanishing effect

$$\partial_{y_1}\partial_{y_2}(\partial_{y_1} - \partial_{y_2}) D_z(y_1, y_2) \int^z dy \tilde{\rho}(y) = 2(y_1 - y_2), \tag{4.10}$$

which is the reason why the naive extension violates the integrability condition. However, $\tilde{\rho}$ is not a pure c-number and contains the operator part $\mathcal{O}_0$. Then, on making the substitution as in the $c = 0$ case, the algebra of $\partial_y T_0(y)$ becomes

$$\begin{aligned} [\partial_{y_1} T(y_1), \partial_{y_2} T(y_2)] &= -g_{\rm st}^2 \partial_{y_1}\partial_{y_2}(\partial_{y_1} - \partial_{y_2}) \frac{1}{y_1 - y_2} (T(y_1) - T(y_2)) \\ &\quad + 2g_{\rm st}^2 (y_1 - y_2) \\ &\quad + \partial_{y_1} [T_0(y_1), \mathcal{O}_0] - \partial_{y_2} [T_0(y_2), \mathcal{O}_0]. \end{aligned} \tag{4.11}$$

Since $\mathcal{O}_0$ inserts a microscopic loop, it can be expressed by some local operator, obtained as some coefficient of large $y$ expansion of the loop operator $\frac{\delta}{\delta \tilde{J}(y)}$. This is done in the Appendix B. The final results are

$$\mathcal{O}_0 = 2 \int_C \frac{dy}{2\pi i} y^{1/2} \frac{\delta}{\delta \tilde{J}(y)} \tag{4.12}$$

$$= 2 \lim_{\varepsilon \to +0} \int_{-i\infty}^{i\infty} \frac{dy}{2\pi i} \mathrm{e}^{\varepsilon y} y^{1/2} \frac{\delta}{\delta \tilde{J}(y)}, \tag{4.13}$$

where the contour $C$ surrounds the negative real axis, and the contour $[-i\infty, i\infty]$ is understood to avoid the singularity at the origin to the right. Also, the integral is defined by the

analytic continuation using the Beta function and the limit $\varepsilon \to +0$ must be taken **after** the integration.

Now in calculating the commutator $[T_0(y), \mathcal{O}_0]$, we can use (4.13) rather than (4.12), because the functional derivative

$$\frac{\delta \tilde{J}(y)}{\delta \tilde{J}(y')} = 2\pi i \delta(y - y')$$

is defined for $y, y'$ on the imaginary axis. As a result of the straightforward calculation performing partial integration once, the commutator $[T_0(y), \mathcal{O}_0]$ becomes

$$[T_0(y), \mathcal{O}_0] = -g_{\rm st}^2 \int_C \frac{dy'}{2\pi i} y'^{-1/2} \frac{1}{y' - y} \left( \frac{\delta}{\delta \tilde{J}(y')} - \frac{\delta}{\delta \tilde{J}(y)} \right). \tag{4.14}$$

Substituting (B.20), and using the formulas in the Appendix B, we see that only the term

$$-g_{\rm st}^2 y^{5/2} \int_C \frac{dy'}{2\pi i} y'^{-1/2} \frac{1}{y - y'}$$

survives after some cancellations. Thus, we obtain

$$[T_0(y), \mathcal{O}_0] = -g_{\rm st}^2 y^2, \tag{4.15}$$

which makes the algebra (4.11) for $T(y)$ closed:

$$[\partial_{y_1} T(y_1), \partial_{y_2} T(y_2)] = -g_{\rm st}^2 \partial_{y_1} \partial_{y_2} (\partial_{y_1} - \partial_{y_2}) \frac{1}{y_1 - y_2} (T(y_1) - T(y_2)). \tag{4.16}$$

We note that this algebra is of course identical with the usual Virasoro algebra (B.18), after taking into account the contribution from the transformation (B.19). All what we have done is merely a check of self consistency. It clarifies, however, how the closure of the Schwinger-Dyson operators associated with the Hamiltonian is satisfied for higher critical points, owing to the presence of the operator part of $\tilde{\rho}(y)$.

## 5. $c = 1/2$ PSFT from Two-Matrix Model

We now apply our method to the two-matrix model and derive the $c = 1/2$ PSFT without making any restrictions on the spin configurations of the string fields. As we emphasized in the Introduction, such a treatment will hopefully reveal certain universal properties of the PSFT which are basically independent of the structure of target spaces. This is our motivation for performing this analysis in spite of its technical difficulties.

*5.1 Stochastic Hamiltonian of the Two-Matrix Model*

The generating functional of the two-matrix model is defined by

$$\begin{aligned}
Z[J] &= \frac{1}{Z}\int d^{N^2}A d^{N^2}B\,\mathrm{e}^{-S}\mathrm{e}^{J\cdot\Phi},\\
Z &= \int d^{N^2}A d^{N^2}B\,\mathrm{e}^{-S},\\
S &= N\mathrm{tr}(V(A)+V(B)-cAB), \quad V(A)=\frac{1}{2}A^2-\frac{g}{3}A^3,
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
J\cdot\Phi &= \int_L \frac{d\zeta}{2\pi i}J_A(\zeta)\Phi_A(\zeta)+\int_L \frac{d\sigma}{2\pi i}J_B(\sigma)\Phi_B(\sigma)\\
&+\sum_{n=1}^{\infty}\int_L\prod_{i=1}^{n}\frac{d\zeta_i}{2\pi i}\frac{d\sigma_i}{2\pi i}J_n(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n)\Phi_n(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n),
\end{aligned} \tag{5.2}$$

where the components of the string field $\Phi$ are defined in terms of the matrix variables as

$$\begin{aligned}
\Phi_A(\zeta) &= \frac{1}{N}\mathrm{tr}\frac{1}{\zeta-A},\\
\Phi_B(\sigma) &= \frac{1}{N}\mathrm{tr}\frac{1}{\sigma-B},\\
\Phi_n(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n) &= \frac{1}{N}\mathrm{tr}\left(\frac{1}{\zeta_1-A}\frac{1}{\sigma_1-B}\cdots\frac{1}{\zeta_n-A}\frac{1}{\sigma_n-B}\right)\\
&\qquad (n=1,2,\cdots).
\end{aligned}$$

The component $\Phi_A$ ($\Phi_B$) represents a loop on which only a single spin $A$ ($B$) is put, and $\Phi_n$ represents a loop on which two domains of $A$- and $B$-spins alternatively appear $n$ times. The variables $\zeta_i,\sigma_i \quad (i=1,2,\ldots,n)$ can be regarded as the conjugate variables corresponding to lengths on the loop for spinstates $A, B$, respectively.

As before, the stochastic Hamiltonian for this system is derived from the identity

$$0=-\int d^{N^2}A d^{N^2}B\sum_{\alpha=1}^{N^2}\left[\frac{\partial}{\partial A_\alpha}\mathrm{e}^{-S}\frac{\partial}{\partial A_\alpha}+\frac{\partial}{\partial B_\alpha}\mathrm{e}^{-S}\frac{\partial}{\partial B_\alpha}\right]\mathrm{e}^{J\cdot\Phi}. \tag{5.3}$$

By extending the formulas for the one-matrix model, we can arrange eq. (5.3) in the following form:

$$\begin{aligned}
0 &= \mathcal{H}Z[J]\\
\mathcal{H} &= -J\cdot\left(K\frac{\delta}{\delta J}\right)-J\cdot\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)-\frac{1}{N^2}J\cdot\left(J\cdot\left(\wedge\frac{\delta}{\delta J}\right)\right)-J\cdot T.
\end{aligned} \tag{5.4}$$

Here, reflecting the cyclic symmetry of pairs $(\zeta_i,\sigma_i)$, the derivative $\frac{\delta}{\delta J_n}$ is defined by

$$\begin{aligned}
&\frac{\delta J_m(\zeta_1',\sigma_1',\cdots,\zeta_m',\sigma_m')}{\delta J_n(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n)}=\delta_{m,n}\frac{1}{n}(2\pi i)^{2n}\\
&\times\sum_{c:\text{cyclic permutation}}\delta(\zeta_1-\zeta_{c(1)}')\delta(\sigma_1-\sigma_{c(1)}')\cdots\delta(\zeta_n-\zeta_{c(n)}')\delta(\sigma_n-\sigma_{c(n)}').
\end{aligned} \tag{5.5}$$

Let us explain the meaning of each term.
i) The first term, "kinetic term", $J \cdot \left(K \frac{\delta}{\delta J}\right)$, coming from a part of the product of the derivatives of the classical action and the source term, symbolizes the contributions which preserve the number of string fields. The first few components of the kinetic operator $K$ are as follows.

$$\begin{aligned}
\left(K \frac{\delta}{\delta J}\right)_A (\zeta) &= \vec{\partial}_\zeta (\zeta - g\zeta^2) \frac{\delta}{\delta J_A(\zeta)} - c\partial_\zeta \oint \frac{d\sigma}{2\pi i} \sigma \frac{\delta}{\delta J_1(\zeta, \sigma)}, \\
\left(K \frac{\delta}{\delta J}\right)_B (\sigma) &= \vec{\partial}_\sigma (\sigma - g\sigma^2) \frac{\delta}{\delta J_B(\sigma)} - c\partial_\sigma \oint \frac{d\zeta}{2\pi i} \zeta \frac{\delta}{\delta J_1(\zeta, \sigma)},
\end{aligned}$$

$$\begin{aligned}
\left(K \frac{\delta}{\delta J}\right)_1 (\zeta_1, \sigma_1) &= (\vec{\partial}_{\zeta_1} (\zeta_1 - g\zeta_1^2) + \vec{\partial}_{\sigma_1} (\sigma_1 - g\sigma_1^2)) \frac{\delta}{\delta J_1(\zeta_1, \sigma_1)} \\
&+ c \oint \frac{d\sigma}{2\pi i} \sigma \frac{\delta}{\delta J_2(\zeta_1, \sigma, \zeta_1, \sigma_1)} + c \oint \frac{d\zeta}{2\pi i} \zeta \frac{\delta}{\delta J_2(\zeta_1, \sigma_1, \zeta, \sigma_1)} \\
&+ g \left( \frac{\delta}{\delta J_A(\zeta_1)} + \frac{\delta}{\delta J_B(\sigma_1)} \right),
\end{aligned}$$

$$\begin{aligned}
\left(K \frac{\delta}{\delta J}\right)_2 (\zeta_1, \sigma_1, \zeta_2, \sigma_2) &= \sum_{j=1}^{2} \{ \vec{\partial}_{\zeta_j} (\zeta_j - g\zeta_j^2) + \vec{\partial}_{\sigma_j} (\sigma_j - g\sigma_j^2) \} \frac{\delta}{\delta J_2(\zeta_1, \sigma_1, \zeta_2, \sigma_2)} \\
&+ c \oint \frac{d\sigma}{2\pi i} \sigma \left( \frac{\delta}{\delta J_3(\zeta_1, \sigma, \zeta_1, \sigma_1, \zeta_2, \sigma_2)} + \frac{\delta}{\delta J_3(\zeta_1, \sigma_1, \zeta_2, \sigma, \zeta_2, \sigma_2)} \right) \\
&+ c \oint \frac{d\zeta}{2\pi i} \zeta \left( \frac{\delta}{\delta J_3(\zeta_1, \sigma_1, \zeta, \sigma_1, \zeta_2, \sigma_2)} + \frac{\delta}{\delta J_3(\zeta_1, \sigma_1, \zeta_2, \sigma_2, \zeta, \sigma_2)} \right) \\
&- g D_\sigma(\sigma_1, \sigma_2) \left( \frac{\delta}{\delta J_1(\zeta_1, \sigma)} + \frac{\delta}{\delta J_1(\zeta_2, \sigma)} \right) \\
&- g D_\zeta(\zeta_1, \zeta_2) \left( \frac{\delta}{\delta J_1(\zeta, \sigma_1)} + \frac{\delta}{\delta J_1(\zeta, \sigma_2)} \right), \\
&\cdots, \qquad (5.6)
\end{aligned}$$

where the arrow over $\partial$ indicates that the derivative acts on the whole functions that follow it.

The structure of the higher components can be inferred from these expressions. Basically, each component represents one of the following two processes. The first is the propagation of string preserving a spin configuration on a loop, with the loop length being either kept fixed or decreased by one-lattice unit. The second is a process in which only a single spin is flipped and the loop length is preserved. For example, let us consider $\left(K \frac{\delta}{\delta J}\right)_1 (\zeta_1, \sigma_1)$. The first and last terms express the former process. Note that, as a special case when a domain consists of only one spin, the process can annihilate the domain. The last term represents

this. On the other hand, the second and third terms express the latter process, with a single spin flip preserving the loop length.

In the case of $\left(K\frac{\delta}{\delta J}\right)_2$, it is noted that $-D_\sigma(\sigma_1,\sigma_2)\frac{\delta}{\delta J_1(\zeta_1,\sigma)}$ in $\left(K\frac{\delta}{\delta J}\right)_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)$ represents the following loop when it acts on the partition function,

$$-D_\sigma(\sigma_1,\sigma_2)\frac{\delta}{\delta J_1(\zeta_1,\sigma)} = \frac{1}{N}\mathrm{tr}\left(\frac{1}{\zeta_1 - A}\frac{1}{\sigma_1 - B}\frac{1}{\sigma_2 - B}\right).$$

Namely, the $\zeta_2$-domain has disappeared in $\Phi_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)$.

ii) The second term $J\cdot\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)$, coming from the second derivative of the source term, represents processes where a string splits into two. The symbol $\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)_I$ represents the result of splitting of a string with the spin configuration $I$. The first few components are

$$\begin{aligned}
\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)_A(\zeta) &= -\partial_\zeta\frac{\delta^2}{\delta J_A(\zeta)^2},\\
\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)_B(\sigma) &= -\partial_\sigma\frac{\delta^2}{\delta J_B(\sigma)^2},\\
\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)_1(\zeta_1,\sigma_1) &= -2\left(\frac{\delta}{\delta J_A(\zeta_1)}\partial_{\zeta_1}+\frac{\delta}{\delta J_B(\sigma_1)}\partial_{\sigma_1}\right)\frac{\delta}{\delta J_1(\zeta_1,\sigma_1)},\\
\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2) &= -2\sum_{j=1}^{2}\left(\frac{\delta}{\delta J_A(\zeta_j)}\partial_{\zeta_j}+\frac{\delta}{\delta J_B(\sigma_j)}\partial_{\sigma_j}\right)\frac{\delta}{\delta J_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)}\\
&\quad +2D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta J_1(\zeta,\sigma_1)}D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta J_1(\zeta,\sigma_2)}\\
&\quad +2D_\sigma(\sigma_1,\sigma_2)\frac{\delta}{\delta J_1(\zeta_1,\sigma)}D_\sigma(\sigma_1,\sigma_2)\frac{\delta}{\delta J_1(\zeta_2,\sigma)},\\
&\cdots.
\end{aligned}\tag{5.7}$$

We hope the structure of the higher terms containing is self-explanatory from these examples.
iii) The third term $\frac{1}{N^2}J\cdot\left(J\cdot\left(\wedge\frac{\delta}{\delta J}\right)\right)$, coming from the square of the first derivative of the source term, symbolizes processes in which two strings merge into a single string. The symbol $\left(\wedge\frac{\delta}{\delta J}\right)_{IJ}$ expresses the result of the two strings with the spin configurations $I, J$ merging into a single string :

$$\begin{aligned}
\left(\wedge\frac{\delta}{\delta J}\right)_{A,A}(\zeta;\zeta') &= -\partial_\zeta\partial_{\zeta'}D_z(\zeta,\zeta')\frac{\delta}{\delta J_A(z)},\\
\left(\wedge\frac{\delta}{\delta J}\right)_{B,B}(\sigma;\sigma') &= -\partial_\sigma\partial_{\sigma'}D_s(\sigma,\sigma')\frac{\delta}{\delta J_B(s)},\\
\left(\wedge\frac{\delta}{\delta J}\right)_{A,B}(\zeta;\sigma) &= 0,
\end{aligned}$$

$$\left(\wedge\frac{\delta}{\delta J}\right)_{n,A}(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n;\zeta') = \left(\wedge\frac{\delta}{\delta J}\right)_{A,n}(\zeta';\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n)$$
$$= \ -\partial_{\zeta'}\sum_{j=1}^{n}\partial_{\zeta_j}D_z(\zeta_j,\zeta')\frac{\delta}{\delta J_n(\zeta_1,\sigma_1,\cdots,z,\sigma_j,\cdots,\zeta_n,\sigma_n)},$$

$$\left(\wedge\frac{\delta}{\delta J}\right)_{n,B}(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n;\sigma') = \left(\wedge\frac{\delta}{\delta J}\right)_{B,n}(\sigma';\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n)$$
$$= \ -\partial_{\sigma'}\sum_{j=1}^{n}\partial_{\sigma_j}D_s(\sigma_j,\sigma')\frac{\delta}{\delta J_n(\zeta_1,\sigma_1,\cdots,\zeta_j,s,\cdots,\zeta_n,\sigma_n)},$$

$$\left(\wedge\frac{\delta}{\delta J}\right)_{n,1}(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n;\zeta_1',\sigma_1') = \left(\wedge\frac{\delta}{\delta J}\right)_{1,n}(\zeta_1',\sigma_1';\zeta_1,\sigma_1,\cdots,\zeta_n.\sigma_n)$$
$$= \ \sum_{j=1}^{n}D_z(\zeta_j,\zeta_1')D_w(\zeta_j,\zeta_1')\frac{\delta}{\delta J_{n+1}(\zeta_1,\sigma_1,\cdots,\zeta_{j-1},\sigma_{j-1},z,\sigma_1',w,\sigma_j,\cdots,\zeta_n,\sigma_n)}$$
$$+ \ \sum_{j=1}^{n}D_s(\sigma_j,\sigma_1')D_t(\sigma_j,\sigma_1')\frac{\delta}{\delta J_{n+1}(\zeta_1,\sigma_1,\cdots,\zeta_j,s,\zeta_1',t,\zeta_{j+1},\sigma_{j+1},\cdots,\zeta_n,\sigma_n)},$$

$$\cdots. \qquad\qquad (n=1,2,\cdots) \tag{5.8}$$

We again hope that the structure of the generic term is self-explanatory from these examples. iv) The last term (the tadpole term), which together with the kinetic term is originated from the product of the derivatives of the classical action and the source term, shows the processes of the annihilation of a string into nothing:

$$J\cdot T = \int_L \frac{d\zeta}{2\pi i}J_A(\zeta)g + \int_L \frac{d\sigma}{2\pi i}J_B(\sigma)g.$$

We here would like to emphasize an important property which characterizes all of the above formulas and plays an essential role later in studying the scaling limit. Namely, all the processes occur **locally** with respect to the spin domains. Because of locality, more than two domains never be created or annihilated at the same time. As a consequence of this rule, only the strings consisting of one domain, $\Phi_A$ and $\Phi_B$, can be annihilated into nothing. Also, only a single pair of domains can participate in the splitting or merging processes, and other domains are left intact.

### *5.2 Hamiltonian in Continuum Theory*

Let us now consider the continuum limit of the Hamiltonian (5.4). As in the one-matrix cases, the first task in carrying out this is to identify and to subtract the non-universal parts of the disk amplitudes. At this point, a new feature arises. Namely, as we discuss in the Appendix

C, the non-universal part of a disk amplitude with a given spin configuration contains, in general, the universal parts of the disk amplitudes with simpler spin configurations, in addition to the non-universal c-number function which has already appeared in the one-matrix model.

By introducing the connected $k$-point correlator in the $J=0$ background as

$$G^{(k)}_{I_1,\cdots,I_k} = \langle \Phi_{I_1} \cdots \Phi_{I_k} \rangle, \qquad I_1,\cdots,I_k = A,B,1,2,\cdots, \tag{5.9}$$

the generating functional is written as

$$Z[J] = \exp\left[ J\cdot G^{(1)} + \frac{1}{2!} J\cdot (J\cdot G^{(2)}) + \frac{1}{3!} J\cdot (J\cdot (J\cdot G^{(3)})) + \cdots \right]. \tag{5.10}$$

Now, the investigation of disk amplitudes in the Appendix C shows that the universal part $\hat{\Phi}_I$ of the operator $\Phi_I$ is obtained by a linear transformation of the following form:

$$\Phi_I = \sum_J \mathcal{M}_{IJ} \hat{\Phi}_J + \phi_I, \tag{5.11}$$

where $\mathcal{M}_{IJ}$ is a mixing matrix of the universal parts, which is the new feature noticed above, and $\phi_I$ is the non-universal c-number function. The mixing matrix $\mathcal{M}_{IJ}$ is upper-triangular, i.e. $\mathcal{M}_{IJ}=0$ for $I<J$ and is invertible. The first few components of (5.11) are

$$\begin{aligned}
\Phi_A(\zeta) &= \hat{\Phi}_A(\zeta) + \phi_A(\zeta), \\
\Phi_B(\sigma) &= \hat{\Phi}_B(\sigma) + \phi_B(\sigma), \\
\Phi_1(\zeta,\sigma) &= \sqrt{10c}(\hat{\Phi}_A(\zeta) + \hat{\Phi}_B(\sigma)) + \hat{\Phi}_1(\zeta,\sigma) + \phi_1(\zeta,\sigma), \\
\Phi_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2) &= -10c(D_\zeta(\zeta_1,\zeta_2)\hat{\Phi}_A(\zeta) + D_\sigma(\sigma_1,\sigma_2)\hat{\Phi}_B(\sigma)) \\
&\quad -\sqrt{10c}[D_\zeta(\zeta_1,\zeta_2)(\hat{\Phi}_1(\zeta,\sigma_1) + \hat{\Phi}_1(\zeta,\sigma_2)) \\
&\quad\ + D_\sigma(\sigma_1,\sigma_2)(\hat{\Phi}_1(\zeta_1,\sigma) + \hat{\Phi}_1(\zeta_2,\sigma))] \\
&\quad +\hat{\Phi}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2) + \phi_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2), \\
&\cdots,
\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}
\phi_A(\zeta) &= -\frac{c}{3g} + \frac{2}{3}(\zeta - g\zeta^2), \\
\phi_B(\sigma) &= -\frac{c}{3g} + \frac{2}{3}(\sigma - g\sigma^2), \\
\phi_1(\zeta,\sigma) &= c(1+2s-\sqrt{10c}(\zeta+\sigma)), \\
\phi_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2) &= 10c^2, \\
&\cdots,
\end{aligned} \tag{5.13}$$

and $c$ takes its critical value: $c = \frac{-1+2\sqrt{7}}{27}$, and $s = 2 + \sqrt{7}$. Then the connected correlators are transformed as

$$
\begin{aligned}
G^{(1)}_I &= \sum_I \mathcal{M}_{IJ} \hat{G}^{(1)}_J + \phi_I, \\
G^{(k)}_{I_1,\cdots,I_k} &= \sum_{J_1,\cdots,J_k} \mathcal{M}_{I_1 J_1} \cdots \mathcal{M}_{I_k J_k} \hat{G}^{(k)}_{J_1,\cdots,J_k} \quad (k \geq 2),
\end{aligned}
$$

where $\hat{G}^{(k)}_{I_1,\cdots,I_k}$ stands for the universal part of $G^{(k)}_{I_1,\cdots,I_k}$.

Thus, by introducing the transformed source

$$
J_I = \sum_K \hat{J}_K (\mathcal{M}^{-1})_{KI}, \tag{5.14}
$$

the generating functional $\hat{Z}[\hat{J}]$ in the continuum theory is obtained by the rescaling $Z[J] = \mathrm{e}^{J\cdot\phi} \hat{Z}[\hat{J}]$, and takes the form

$$
\hat{Z}[\hat{J}] = \exp\left[\hat{J} \cdot \hat{G}^{(1)} + \frac{1}{2!}\hat{J} \cdot (\hat{J} \cdot \hat{G}^{(2)}) + \frac{1}{3!}\hat{J} \cdot (\hat{J} \cdot (\hat{J} \cdot \hat{G}^{(3)})) + \cdots\right]. \tag{5.15}
$$

The Hamiltonian acting on $\hat{Z}[\hat{J}]$ now becomes

$$
\begin{aligned}
0 &= \mathcal{H} \hat{Z}[\hat{J}], \qquad (5.16)\\
\mathcal{H} &= -(\hat{J}\mathcal{M}^{-1}) \cdot \left(K(\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi)\right) - (\hat{J}\mathcal{M}^{-1}) \cdot T \\
&\quad -(\hat{J}\mathcal{M}^{-1}) \cdot \left((\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi) \vee (\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi)\right) \\
&\quad -\frac{1}{N^2}(\hat{J}\mathcal{M}^{-1}) \cdot \left((\hat{J}\mathcal{M}^{-1}) \cdot \left(\wedge(\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi)\right)\right). \qquad (5.17)
\end{aligned}
$$

Next, we arrange eq. (5.17) to a simpler form in which the mixing matrix disappears. We claim the validity of the following equations:

$$
\begin{aligned}
&-(\hat{J}\mathcal{M}^{-1}) \cdot \left(K(\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi)\right) - (\hat{J}\mathcal{M}^{-1}) \cdot T \\
&\quad -(\hat{J}\mathcal{M}^{-1}) \cdot \left((\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi) \vee (\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi)\right) \\
&= -\hat{J} \cdot \left(F\frac{\delta}{\delta \hat{J}}\right) - (\hat{J}\mathcal{M}^{-1}) \cdot \left(\left(\mathcal{M}\frac{\delta}{\delta \hat{J}}\right) \vee \left(\mathcal{M}\frac{\delta}{\delta \hat{J}}\right)\right), \qquad (5.18)
\end{aligned}
$$

$$
\begin{aligned}
(\hat{J}\mathcal{M}^{-1}) \cdot \left(\left(\mathcal{M}\frac{\delta}{\delta \hat{J}}\right) \vee \left(\mathcal{M}\frac{\delta}{\delta \hat{J}}\right)\right) &= \hat{J} \cdot \left(\frac{\delta}{\delta \hat{J}} \vee \frac{\delta}{\delta \hat{J}}\right), \qquad (5.19)\\
(\hat{J}\mathcal{M}^{-1}) \cdot \left((\hat{J}\mathcal{M}^{-1}) \cdot \left(\wedge \mathcal{M}\frac{\delta}{\delta \hat{J}}\right)\right) &= \hat{J} \cdot \left(\hat{J} \cdot \left(\wedge \frac{\delta}{\delta \hat{J}}\right)\right), \qquad (5.20)\\
(\hat{J}\mathcal{M}^{-1}) \cdot ((\hat{J}\mathcal{M}^{-1}) \cdot (\wedge \phi)) &= 0, \qquad (5.21)
\end{aligned}
$$

where $F$ is a part of the kinetic operator $K$, representing only the spin flip process. These equations make possible to rewrite the Hamiltonian as

$$\mathcal{H} = -\hat{J} \cdot \left( F \frac{\delta}{\delta \hat{J}} \right) - \hat{J} \cdot \left( \frac{\delta}{\delta \hat{J}} \vee \frac{\delta}{\delta \hat{J}} \right) - \frac{1}{N^2} \hat{J} \cdot \left( \hat{J} \cdot \left( \wedge \frac{\delta}{\delta \hat{J}} \right) \right). \tag{5.22}$$

**Justification of the equations (5.18)∼(5.21)** :

We now present the arguments for establishing the above equations. Firstly, we consider the spin flip process in the continuum theory. For the universal parts of the disk amplitudes, the loop containing a microscopic domain which consists only of a single flipped spin is obtained from the loop containing only macroscopic domains by the following rule:

$$\begin{aligned} -\partial_\zeta \hat{W}_1(\zeta) &= -s^{-1} \hat{\oint} \frac{d\sigma}{2\pi i} \sigma \partial_\zeta \hat{W}^{(2)}(\zeta, \sigma), && (5.23) \\ \hat{W}_1(\zeta_1; \zeta_2, \sigma_1) &= s^{-1} \hat{\oint} \frac{d\sigma}{2\pi i} \sigma \hat{W}^{(4)}(\zeta_1, \sigma, \zeta_2, \sigma_1), && (5.24) \\ &\cdots, \end{aligned}$$

where the domain $\sigma$ has been shrunk into the microscopic domain by integration. Here we borrow the notations of ref. [20] for the disk amplitudes. For their definitions, see the Appendix C. The integral symbol $\hat{\oint} \frac{d\sigma}{2\pi i} \sigma$ is used in the sense of the integral of the variable $x$ in the continuum theory ($\sigma = P_*(1 + ax)$)

$$\hat{\oint} \frac{d\sigma}{2\pi i} \sigma = P_*^2 a \int_C \frac{dx}{2\pi i}, \tag{5.25}$$

where the contour $C$ encircles around the negative real axis and the singularities of the left half plane. $P_*$ stands for the critical value $P_* = s(10c)^{-1/2}$, and $a$ is a lattice spacing. Note that there is a sort of finite *renormalization* represented by the factor $s^{-1}$. The derivation of the above formulas is given in the Appendix D.

These relations reflect the property that the spin flip process occurs locally with respect to domains; in (5.23), (5.24) only the $\sigma$ domain is concerned and the other domains do not change at all. This implies that the relation such as (5.23) and (5.24) should hold for any amplitudes with arbitrary number of handles with generic spin configurations. The complete general proof of this important property would be, however, technically formidable, since it would require explicit identification of the universal part for general loops with arbitrary spin configurations. We can now rewrite the spin flip process in $\left( K \left( \mathcal{M} \frac{\delta}{\delta \hat{J}} + \phi \right) \right)_A (\zeta)$ as,

$$\begin{aligned} &-\partial_\zeta \oint \frac{d\sigma}{2\pi i} \sigma [ \left( \mathcal{M} \frac{\delta}{\delta \hat{J}} \right)_1 (\zeta, \sigma) + \phi_1(\zeta, \sigma)] \\ &= \ -s^{-1} \hat{\oint} \frac{d\sigma}{2\pi i} \sigma \partial_\zeta \frac{\delta}{\delta \hat{J}_1(\zeta, \sigma)} - \partial_\zeta \left( \left( \frac{2}{3g} - \frac{1}{3c}(\zeta - g\zeta^2) \right) \frac{\delta}{\delta \hat{J}_A(\zeta)} \right) \\ &\quad -\frac{1}{c} \partial_\zeta [g\zeta + \frac{c}{9g}(\zeta - g\zeta^2) + \frac{2}{9}(\zeta - g\zeta^2)^2], \end{aligned} \tag{5.26}$$

where the first term is the universal part, and the others are the non-universal parts which are obtained by replacing $\hat{W}(\zeta)$ with $\frac{\delta}{\delta \hat{J}_A(\zeta)}$ in (C.19).

Similarly, we can derive the expressions for

$$-\partial_\sigma \oint \frac{d\zeta}{2\pi i}\zeta[\left(\mathcal{M}\frac{\delta}{\delta \hat{J}}\right)_1(\zeta,\sigma)+\phi_1(\zeta,\sigma)] \quad \text{in} \quad \left(K\left(\mathcal{M}\frac{\delta}{\delta \hat{J}}+\phi\right)\right)_B(\sigma),$$

and

$$\oint \frac{d\sigma}{2\pi i}\sigma[\left(\mathcal{M}\frac{\delta}{\delta \hat{J}}\right)_2(\zeta_1,\sigma,\zeta_1,\sigma_1)+\phi_2],$$
$$\oint \frac{d\zeta}{2\pi i}\zeta[\left(\mathcal{M}\frac{\delta}{\delta \hat{J}}\right)_2(\zeta_1,\sigma_1,\zeta,\sigma_1)+\phi_2] \quad \text{in} \quad \left(K\left(\mathcal{M}\frac{\delta}{\delta \hat{J}}+\phi\right)\right)_1(\zeta_1,\sigma_1).$$

Using these results, we arrive at eq. (5.18) with

$$\begin{aligned}
\hat{J}\cdot\left(F\frac{\delta}{\delta \hat{J}}\right) &= \int_L \frac{d\zeta}{2\pi i}\hat{J}_A(\zeta)cs^{-1}(-\partial_\zeta)\hat{\oint}\frac{d\sigma}{2\pi i}\sigma\frac{\delta}{\delta \hat{J}_1(\zeta,\sigma)} \\
&\quad + \int_L \frac{d\sigma}{2\pi i}\hat{J}_B(\sigma)cs^{-1}(-\partial_\sigma)\hat{\oint}\frac{d\zeta}{2\pi i}\zeta\frac{\delta}{\delta \hat{J}_1(\zeta,\sigma)} \\
&\quad + \int_L \frac{d\zeta_1}{2\pi i}\frac{d\sigma_1}{2\pi i}\hat{J}_1(\zeta_1,\sigma_1)cs^{-1}\left[\hat{\oint}\frac{d\sigma}{2\pi i}\sigma\frac{\delta}{\delta \hat{J}_2(\zeta_1,\sigma,\zeta_1,\sigma_1)}\right. \\
&\qquad\qquad \left. + \hat{\oint}\frac{d\zeta}{2\pi i}\zeta\frac{\delta}{\delta \hat{J}_2(\zeta_1,\sigma_1,\zeta,\sigma_1)}\right] \\
&\quad + \cdots,
\end{aligned} \tag{5.27}$$

where the ellipsis stands for the terms containing the higher components $\hat{J}_n$ $(n \geq 2)$.

We here note that the tadpole term is cancelled with a contribution of the same form from the kinetic term. This can be regarded as a consequence of our definition of the disk universal part of the single-spin flip amplitude $\hat{W}_1(\zeta)$: Roughly, the kinetic term contains the product of the form $J_A \times \frac{\delta}{\delta J_A^{\text{one-spin flip}}} + (A \to B)$, and hence there is always a freedom of absorbing the tadpole terms, which are polynomials times the sources $J_A$ $(J_B)$, by appropriately defining the universal parts of the spin-flip disk amplitudes and making the shifts for $\frac{\delta}{\delta J_A^{\text{one-spin flip}}}(\frac{\delta}{\delta J_B^{\text{one-spin flip}}})$. In the case of one-matrix model, the spin-flip process is absent so that the tadpole term is directly responsible for determining the disk amplitude.

Although we do not elaborate further on determining the explicit continuum limits for higher components, it is natural, because of the local nature of the spin-flipping process, to suppose that the above expression already indicates the generic structure of the kinetic term, namely the flipping of a single spin with general spin configurations, the absence of the tadpole term and the kinetic term without spin flipping.

Secondly, we consider the splitting and merging processes, eqs. (5.19) and (5.20). In the Appendix E, it is explicitly checked that

$$\left(\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)\vee\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)\right)_I=\left(\mathcal{M}\left(\frac{\delta}{\delta\hat{J}}\vee\frac{\delta}{\delta\hat{J}}\right)\right)_I \tag{5.28}$$

for $I = A, B, 1, 2$, and that

$$\left(\wedge\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)\right)_{I,J}=\sum_{K,L}\mathcal{M}_{IK}\mathcal{M}_{JL}\left(\wedge\frac{\delta}{\delta\hat{J}}\right)_{K,L} \tag{5.29}$$

for $(I, J) = (A, A), (B, B), (A, 1), (B, 1), (A, 2), (B, 2), (1, 1)$.

As seen from eqs. (5.11) and (5.12), $\mathcal{M}$ mixes the operators with the configurations which are smaller parts of the configuration $I$, when it acts on the loop operator $\frac{\delta}{\delta\hat{J}_I}$. Eq. (5.28) (eq. (5.29)) means that the mixing is commutative with the splitting (merging) processes of loops. This result can again be regarded as a consequence of locality of the splitting and merging processes. It is then reasonable to assume that the splitting (merging ) commutes with mixing matrix for arbitrary spin configurations. This assertion is nothing but the claims (5.19), (5.20). Proving this for completely general case is prohibitively difficult in our present technical machinery for treating the double scaling limit. We have to be satisfied with the explicit confirmation of this property for several nontrivial cases as given in the Appendix E.

Thirdly, in regard of eq. (5.21), by direct calculation we can check that

$$(\wedge\phi)_{I,J}=0 \tag{5.30}$$

for $(I, J) = (A, A), (B, B), (A, 1), (B, 1), (A, 2), (B, 2), (1, 1)$. In this case, we can give a general proof of this equation as follows. First we show that $\phi_k$ must be polynomial for general $k$. Suppose that this is valid up to some $k-1$. Then, from Staudacher's recursion equations, as described by (C.21), which relates $W^{(2k)}$ with amplitudes with lower values of $k$, we can see that the part of the numerator for $W^{(2k)}$ consisting only of $\phi$ is a polynomial, because in general the combinatorial derivative of a polynomial is also a polynomial. The denominator, on the other hand, behaves in the scaling limit as

$$P_1-gP_1^2-cQ_1-W(P_1)=-a\frac{c^{1/2}s}{\sqrt{10}}(y_1+x_1)+O(a^{4/3}).$$

Thus, by using the scaled variables $P_i=\zeta_*(1+ay_i), Q_i=\zeta_*(1+ax_i)$, the form of $\phi_k$ can be written as

$$\phi_k=\frac{\text{Polynomial of }(y_1,x_1,\cdots,y_k,x_k)}{y_1+x_1}.$$

However, from the cyclic symmetry of $(P_i, Q_i)$ in $W^{(2k)}(P_1, Q_1, \cdots, P_k, Q_k)$, the denominator $y_1 + x_1$ must be cancelled with the numerator, and thus $\phi_k$ should have the form:

$$\phi_k=\text{Polynomial of }(y_1,x_1,\cdots,y_k,x_k),$$

where the polynomial has the same symmetry as $W^{(2k)}$. Thus, by induction, the $\phi_k$ is a polynomial for general $k$.

Now from the scaling of the universal part $\hat{W}^{(2k)}$

$$\hat{W}^{(2k)}(P_1, Q_1, \cdots, P_k, Q_k) = a^{\frac{7}{3}-\frac{2}{3}k} w^{(2k)}(y_1, x_1, \cdots, y_k, x_k),$$

which is presented in the Appendix C, we expect that the relevant part of $\phi_k$ takes the form

$$\phi_k = \begin{cases} \text{const.} & k = 3 \\ 0 & k \geq 4. \end{cases} \tag{5.31}$$

Since every component of $\wedge\phi$ contains the derivative or the combinatorial derivative, eq. (5.31) leads to

$$(\wedge\phi)_{I,J} = 0 \quad \text{for the general components,} \tag{5.32}$$

as is claimed.

**Continuum Limit** :

After these preparations, we are now ready to take the continuum limit of the stochastic Hamiltonian (5.22). From the scaling behaviors of disk amplitudes presented in the Appendix C, the scaling of various variables are fixed as follows:

$$\begin{aligned}
g &= g_*(1 - a^2 \frac{s^2}{20} T), \quad \zeta = P_*(1 + ay), \quad \sigma = P_*(1 + ax), \\
\frac{1}{N} &= a^{7/3} g_{\text{st}}, \\
\frac{\delta}{\delta \hat{J}_A(\zeta)} &= a^{4/3} P_*^{-1} \frac{\delta}{\delta \tilde{J}_A(y)}, \quad \hat{J}_A(\zeta) = a^{-7/3} \tilde{J}_A(y), \\
\frac{\delta}{\delta \hat{J}_B(\sigma)} &= a^{4/3} P_*^{-1} \frac{\delta}{\delta \tilde{J}_B(x)}, \quad \hat{J}_B(\sigma) = a^{-7/3} \tilde{J}_B(x), \\
\frac{\delta}{\delta \hat{J}_1(\zeta, \sigma)} &= a^{5/3} P_*^{-2} \frac{\delta}{\delta \tilde{J}_1(y, x)}, \quad \hat{J}_1(\zeta, \sigma) = a^{-11/3} \tilde{J}_1(y, x), \\
\frac{\delta}{\delta \hat{J}_2(\zeta_1, \sigma_1, \zeta_2, \sigma_2)} &= a^1 P_*^{-4} \frac{\delta}{\delta \tilde{J}_2(y_1, x_1, y_2, x_2)}, \\
& \quad \hat{J}_2(\zeta_1, \sigma_1, \zeta_2, \sigma_2) = a^{-5} \tilde{J}_2(y_1, x_1, y_2, x_2), \\
& \quad \cdots,
\end{aligned} \tag{5.33}$$

where the critical values are [18, 19]

$$g_* = \sqrt{10c^3}, \quad P_* = s(10c)^{-1/2}.$$

Indeed in the limit $a \to 0$ all the universal contributions in the Hamiltonian start with $O(a^{1/3})$, as it should be for the correct continuum limit. After finite rescaling as

$$\tilde{J}_I \to P_*^{-2} \tilde{J}_I, \quad \frac{\delta}{\delta \tilde{J}_I} \to P_*^2 \frac{\delta}{\delta \tilde{J}_I}, \quad g_{\text{st}}^2 \to P_*^4 g_{\text{st}}^2,$$

and absorbing the overall factor $a^{1/3}$ into the fictitious time, we obtain the continuum Hamiltonian in the form

$$\mathcal{H} = -\tilde{J}\cdot\left(F\frac{\delta}{\delta\tilde{J}}\right) - \tilde{J}\cdot\left(\frac{\delta}{\delta\tilde{J}}\vee\frac{\delta}{\delta\tilde{J}}\right) - g_{\mathrm{st}}^2\tilde{J}\cdot\left(\tilde{J}\cdot\left(\wedge\frac{\delta}{\delta\tilde{J}}\right)\right), \tag{5.34}$$

where the inner product is defined by

$$\begin{aligned} f\cdot g &= \int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}f_A(y)g_A(y) + \int_{-i\infty}^{i\infty}\frac{dx}{2\pi i}f_B(x)g_B(x) \\ &+ \sum_{n=1}^{\infty}\int_{-i\infty}^{i\infty}\prod_{i=1}^{n}\frac{dy_i}{2\pi i}\frac{dx_i}{2\pi i}f_n(y_1,x_1,\cdots,y_n,x_n)g_n(y_1,x_1,\cdots,y_n,x_n). \end{aligned} \tag{5.35}$$

As already emphasized above, only the spin flip process survives in the kinetic term:

$$\begin{aligned} \left(F\frac{\delta}{\delta\tilde{J}}\right)_A(y) &= cs^{-1}(-\partial_y)\int_C\frac{dx}{2\pi i}\frac{\delta}{\delta\tilde{J}_1(y,x)}, \\ \left(F\frac{\delta}{\delta\tilde{J}}\right)_B(x) &= cs^{-1}(-\partial_x)\int_C\frac{dy}{2\pi i}\frac{\delta}{\delta\tilde{J}_1(y,x)}, \\ \left(F\frac{\delta}{\delta\tilde{J}}\right)_1(y_1,x_1) &= cs^{-1}\left[\int_C\frac{dx}{2\pi i}\frac{\delta}{\delta\tilde{J}_2(y_1,x,y_1,x_1)} + \int_C\frac{dy}{2\pi i}\frac{\delta}{\delta\tilde{J}_2(y_1,x_1,y,x_1)}\right], \\ \\ \left(F\frac{\delta}{\delta\tilde{J}}\right)_2&(y_1,x_1,y_2,x_2) \\ = cs^{-1}&\left[\int_C\frac{dx}{2\pi i}\left(\frac{\delta}{\delta\tilde{J}_3(y_1,x,y_1,x_1,y_2,x_2)} + \frac{\delta}{\delta\tilde{J}_3(y_1,x_1,y_2,x,y_2,x_2)}\right)\right. \\ &\left.+\int_C\frac{dy}{2\pi i}\left(\frac{\delta}{\delta\tilde{J}_3(y_1,x_1,y,x_1,y_2,x_2)} + \frac{\delta}{\delta\tilde{J}_3(y_1,x_1,y_2,x_2,y,x_2)}\right)\right], \\ \cdots. \end{aligned} \tag{5.36}$$

The forms of the splitting $\frac{\delta}{\delta\tilde{J}}\vee\frac{\delta}{\delta\tilde{J}}$ and the merging $\wedge\frac{\delta}{\delta\tilde{J}}$ are the same as in the lattice theory, because of the commutativity of the mixing matrix with the processes: The first few components of the splitting term are

$$\begin{aligned} \left(\frac{\delta}{\delta\tilde{J}}\vee\frac{\delta}{\delta\tilde{J}}\right)_A(y) &= -\partial_y\frac{\delta^2}{\delta\tilde{J}_A(y)^2}, \\ \left(\frac{\delta}{\delta\tilde{J}}\vee\frac{\delta}{\delta\tilde{J}}\right)_B(x) &= -\partial_x\frac{\delta^2}{\delta\tilde{J}_B(x)^2}, \\ \left(\frac{\delta}{\delta\tilde{J}}\vee\frac{\delta}{\delta\tilde{J}}\right)_1(y,x) &= -2\left(\frac{\delta}{\delta\tilde{J}_A(y)}\partial_y + \frac{\delta}{\delta\tilde{J}_B(x)}\partial_x\right)\frac{\delta}{\delta\tilde{J}_1(y,x)}, \end{aligned}$$

$$
\begin{aligned}
\left(\frac{\delta}{\delta\tilde{J}}\vee\frac{\delta}{\delta\tilde{J}}\right)_2(y_1,x_1,y_2,x_2) &= -2\sum_{j=1}^{2}\left(\frac{\delta}{\delta\tilde{J}_A(y_j)}\partial_{y_j}+\frac{\delta}{\delta\tilde{J}_B(x_j)}\partial_{x_j}\right)\frac{\delta}{\delta\tilde{J}_2(y_1,x_1,y_2,x_2)} \\
&\quad +2\left(D_y(y_1,y_2)\frac{\delta}{\delta\tilde{J}_1(y,x_1)}\right)\left(D_y(y_1,y_2)\frac{\delta}{\delta\tilde{J}_1(y,x_2)}\right) \\
&\quad +2\left(D_x(x_1,x_2)\frac{\delta}{\delta\tilde{J}_1(y_1,x)}\right)\left(D_x(x_1,x_2)\frac{\delta}{\delta\tilde{J}_1(y_2,x)}\right), \\
&\quad \cdots.
\end{aligned}
\tag{5.37}
$$

Similarly, the merging processes are given as

$$
\begin{aligned}
\left(\wedge\frac{\delta}{\delta\tilde{J}}\right)_{A,A}(y;y') &= -\partial_y\partial_{y'}D_z(y,y')\frac{\delta}{\delta\tilde{J}_A(z)}, \\
\left(\wedge\frac{\delta}{\delta\tilde{J}}\right)_{B,B}(x;x') &= -\partial_x\partial_{x'}D_s(x,x')\frac{\delta}{\delta\tilde{J}_B(s)}, \\
\left(\wedge\frac{\delta}{\delta J}\right)_{A,B}(y;x) &= 0,
\end{aligned}
$$

$$
\begin{aligned}
&\left(\wedge\frac{\delta}{\delta\tilde{J}}\right)_{n,A}(y_1,x_1,\cdots,y_n,x_n;y') = \left(\wedge\frac{\delta}{\delta\tilde{J}}\right)_{A,n}(y';y_1,x_1,\cdots,y_n,x_n) \\
&= \; -\partial_{y'}\sum_{j=1}^{n}\partial_{y_j}D_z(y_j,y')\frac{\delta}{\delta\tilde{J}_n(y_1,x_1,\cdots,z,x_j,\cdots,y_n,x_n)},
\end{aligned}
$$

$$
\begin{aligned}
&\left(\wedge\frac{\delta}{\delta\tilde{J}}\right)_{n,B}(y_1,x_1,\cdots,y_n,x_n;x') = \left(\wedge\frac{\delta}{\delta\tilde{J}}\right)_{B,n}(x';y_1,x_1,\cdots,y_n,x_n) \\
&= \; -\partial_{x'}\sum_{j=1}^{n}\partial_{x_j}D_s(x_j,x')\frac{\delta}{\delta\tilde{J}_n(y_1,x_1,\cdots,y_j,s,\cdots,y_n,x_n)},
\end{aligned}
$$

$$
\begin{aligned}
&\left(\wedge\frac{\delta}{\delta\tilde{J}}\right)_{n,1}(y_1,x_1,\cdots,y_n,x_n;y_1',x_1') = \left(\wedge\frac{\delta}{\delta\tilde{J}}\right)_{1,n}(y_1',x_1';y_1,x_1,\cdots,y_n,x_n) \\
&= \; \sum_{j=1}^{n}D_z(y_j,y_1')D_w(y_j,y_1')\frac{\delta}{\delta\tilde{J}_{n+1}(y_1,x_1,\cdots,y_{j-1},x_{j-1},z,x_1',w,x_j,\cdots,y_n,x_n)} \\
&+ \; \sum_{j=1}^{n}D_s(x_j,x_1')D_t(x_j,x_1')\frac{\delta}{\delta\tilde{J}_{n+1}(y_1,x_1,\cdots,y_j,s,y_1',t,y_{j+1},x_{j+1},\cdots,y_n,x_n)},
\end{aligned}
$$

$$
\cdots. \qquad\qquad (n=1,2,\cdots) \tag{5.38}
$$

We remark that the final Hamiltonian has no tadpole term and no dependence on the cosmological constant $T$. Thus in the two-matrix model case, the cosmological constant should be regarded as an integration constant. This contrasts with what one would naively expect from the result for the one-matrix cases. In section 7, to check consistency of the above results, we will discuss how to obtain a closed set of Schwinger-Dyson equations, leading to the $W_3$ constraints, from this Hamiltonian.

## 6. Closure of the Schwinger-Dyson Algebra

Now we proceed to discuss the algebra of the Schwinger-Dyson operators associated with the Hamiltonian (5.34). Comparing with the case of one-matrix models, this requires a much more intricate analysis, since there are an infinite number of components for the $T$ operators:

$$\begin{aligned}\mathcal{H} &= \int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}_A(y)(-\partial_y T_A^A(y)) + \int_{-i\infty}^{i\infty}\frac{dx}{2\pi i}\tilde{J}_B(x)(-\partial_x T_B^B(x)) \\ &+ \sum_{n=1}^{\infty}\int_{-i\infty}^{i\infty}\prod_{i=1}^{n}\frac{dy_i}{2\pi i}\frac{dx_i}{2\pi i}\tilde{J}_n(y_1,x_1,\cdots,y_n,x_n)T_n(y_1,x_1,\cdots,y_n,x_n),\end{aligned} \tag{6.1}$$

$$\begin{aligned}-\partial_y T_A^A(y) &= -\left(F\frac{\delta}{\delta\tilde{J}}\right)_A(y) - \left(\frac{\delta}{\delta\tilde{J}}\vee\frac{\delta}{\delta\tilde{J}}\right)_A(y) \\ &\quad -g_{\rm st}^2\left(\tilde{J}\cdot\left(\wedge\frac{\delta}{\delta\tilde{J}}\right)\right)_A(y),\end{aligned} \tag{6.2}$$

$$\begin{aligned}-\partial_x T_B^B(x) &= -\left(F\frac{\delta}{\delta\tilde{J}}\right)_B(x) - \left(\frac{\delta}{\delta\tilde{J}}\vee\frac{\delta}{\delta\tilde{J}}\right)_B(x) \\ &\quad -g_{\rm st}^2\left(\tilde{J}\cdot\left(\wedge\frac{\delta}{\delta\tilde{J}}\right)\right)_B(x),\end{aligned} \tag{6.3}$$

$$\begin{aligned}T_n(y_1,x_1,\cdots,y_n,x_n) &= -\left(F\frac{\delta}{\delta\tilde{J}}\right)_n(y_1,x_1,\cdots,y_n,x_n) \\ &- \left(\frac{\delta}{\delta\tilde{J}}\vee\frac{\delta}{\delta\tilde{J}}\right)_n(y_1,x_1,\cdots,y_n,x_n) \\ &- g_{\rm st}^2\left(\tilde{J}\cdot\left(\wedge\frac{\delta}{\delta\tilde{J}}\right)\right)_n(y_1,x_1,\cdots,y_n,x_n). \quad (n=1,2,\cdots)\end{aligned} \tag{6.4}$$

The $T_n$ operators appearing in the Hamiltonian can be regarded as the symmetrized versions of the general Schwinger-Dyson operators $T_n^A, T_n^B$:

$$T_1(y_1,x_1) = T_1^A(y_1;y_1,x_1) + T_1^B(y_1,x_1;x_1),$$

$$
\begin{aligned}
T_2(y_1,x_1,y_2,x_2) \;=\;& T_2^A(y_1;y_1,x_1,y_2,x_2)+T_2^A(y_1,x_1,y_2;y_2,x_2)\\
&+T_2^B(y_1,x_1;x_1,y_2,x_2)+T_2^B(y_1,x_1,y_2,x_2;x_2),\\
&\cdots,
\end{aligned}
\tag{6.5}
$$

where the semicolon in the argument in the right hand side denotes the point where the deformation of a loop occurs in constructing the Schwinger-Dyson equation. For example, $T_1^A(y_1;y_1,x_1)$ ($T_1^B(y_1,x_1;x_1)$) represents a deformation of the loop with one pair of domains by attaching on it any loops which have at least one $A(B)$-domain. The explicit forms of the general Schwinger-Dyson operators $T_n^A, T_n^B$ are

$$
\begin{aligned}
&T_n^A(y_{n+1};y_1,x_1,\cdots,y_n,x_n)\\
&= \left(\frac{\delta}{\delta\tilde J_A(y_1)}+\frac{\delta}{\delta\tilde J_A(y_{n+1})}\right)D_z(y_1,y_{n+1})\frac{\delta}{\delta\tilde J_n(z,x_1,\cdots,y_n,x_n)}\\
&- \sum_{j=1}^{n-1}D_z(y_1,y_{j+1})\frac{\delta}{\delta\tilde J_j(z,x_1,\cdots,y_j,x_j)}D_w(y_{j+1},y_{n+1})\frac{\delta}{\delta\tilde J_{n-j}(w,x_{j+1},\cdots,y_n,x_n)}\\
&- cs^{-1}\int_C\frac{dx'}{2\pi i}\frac{\delta}{\delta\tilde J_{n+1}(y_{n+1},x',y_1,x_1,\cdots,y_n,x_n)}\\
&+ g_{\mathrm{st}}^2\int_{-i\infty}^{i\infty}\frac{dy'}{2\pi i}\tilde J_A(y')\partial_{y'}D_z(y_{n+1},y')D_w(z,y_1)\frac{\delta}{\delta\tilde J_n(w,x_1,\cdots,y_n,x_n)}\\
&- g_{\mathrm{st}}^2\sum_{m=1}^{\infty}\int_{-i\infty}^{i\infty}\prod_{i=1}^{m}\frac{dy_i'}{2\pi i}\frac{dx_i'}{2\pi i}\tilde J_m(y_1',x_1',\cdots,y_m',x_m')\sum_{j=1}^{m}D_z(y_j',y_1)D_w(y_j',y_{n+1})\\
&\quad\times\frac{\delta}{\delta\tilde J_{n+m}(y_1',x_1',\cdots,y_{j-1}',x_{j-1}',z,x_1,\cdots,y_n,x_n,w,x_j',y_{j+1}',x_{j+1}',\cdots,y_m',x_m')},
\end{aligned}
\tag{6.6}
$$

$$
\begin{aligned}
&T_n^B(y_1,x_1,\cdots,y_n,x_n;x_{n+1})\\
&= \left(\frac{\delta}{\delta\tilde J_B(x_n)}+\frac{\delta}{\delta\tilde J_B(x_{n+1})}\right)D_z(x_n,y_{n+1})\frac{\delta}{\delta\tilde J_n(y_1,x_1,\cdots,y_n,z)}\\
&- \sum_{j=1}^{n-1}D_z(x_{n+1},x_j)\frac{\delta}{\delta\tilde J_j(y_1,x_1,\cdots,y_j,z)}D_w(x_j,x_n)\frac{\delta}{\delta\tilde J_{n-j}(y_{j+1},x_{j+1},\cdots,y_n,w)}\\
&- cs^{-1}\int_C\frac{dy'}{2\pi i}\frac{\delta}{\delta\tilde J_{n+1}(y_1,x_1,\cdots,y_n,x_n,y',x_{n+1})}\\
&+ g_{\mathrm{st}}^2\int_{-i\infty}^{i\infty}\frac{dx'}{2\pi i}\tilde J_B(x')\partial_{x'}D_z(x_n,x')D_w(z,x_{n+1})\frac{\delta}{\delta\tilde J_n(y_1,x_1,\cdots,y_n,w)}\\
&- g_{\mathrm{st}}^2\sum_{m=1}^{\infty}\int_{-i\infty}^{i\infty}\prod_{i=1}^{m}\frac{dy_i'}{2\pi i}\frac{dx_i'}{2\pi i}\tilde J_m(y_1',x_1',\cdots,y_m',x_m')\sum_{j=1}^{m}D_z(x_j',x_{n+1})D_w(x_j',x_n)\\
&\quad\times\frac{\delta}{\delta\tilde J_{n+m}(y_1',x_1',\cdots,y_{j-1}',x_{j-1}',y_j',z,y_1,x_1,\cdots,y_n,w,y_{j+1}',x_{j+1}',\cdots,y_m',x_m')}.
\end{aligned}
$$

(6.7)

It should be noted that the Hamiltonian contains the particular set of the above operators $T_n^A, T_n^B$, with partial identification of the variables as $y_{n+1} = y_1$, $x_n = x_{n+1}$, respectively.

We also remark that, as is expected from our construction of the Hamiltonian, the general Schwinger-Dyson operators introduced above are the continuum versions of the Schwinger-Dyson operators of the original two-matrix model, appearing in

$$0 = \int d^{N^2}A d^{N^2}B\ T^{(\mathrm{mat.})}(\zeta, \cdots)\mathrm{e}^{-S+J\cdot\Phi}. \tag{6.8}$$

The explicit forms of the matrix model operators are

$$\begin{aligned} T_A^{(\mathrm{mat.})\,A}(\zeta) &= -\frac{1}{N^2}\sum_{\alpha=1}^{N^2}\frac{\vec{\partial}}{\partial A_\alpha}\mathrm{tr}\left(\frac{1}{\zeta - A}t^\alpha\right), \\ T_B^{(\mathrm{mat.})\,B}(\sigma) &= -\frac{1}{N^2}\sum_{\alpha=1}^{N^2}\frac{\vec{\partial}}{\partial B_\alpha}\mathrm{tr}\left(\frac{1}{\sigma - B}t^\alpha\right), \end{aligned}$$

$$\begin{aligned} &T_n^{(\mathrm{mat.})\,A}(\zeta_{n+1};\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n) \\ &= -\frac{1}{N^2}\sum_{\alpha=1}^{N^2}\frac{\vec{\partial}}{\partial A_\alpha}\mathrm{tr}\left(\frac{1}{\zeta_{n+1}-A}t^\alpha\frac{1}{\zeta_1 - A}\frac{1}{\sigma_1 - B}\cdots\frac{1}{\zeta_n - A}\frac{1}{\sigma_n - B}\right), \end{aligned}$$

$$\begin{aligned} &T_n^{(\mathrm{mat.})\,B}(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n;\sigma_{n+1}) \\ &= -\frac{1}{N^2}\sum_{\alpha=1}^{N^2}\frac{\vec{\partial}}{\partial B_\alpha}\mathrm{tr}\left(\frac{1}{\zeta_1 - A}\frac{1}{\sigma_1 - B}\cdots\frac{1}{\zeta_n - A}\frac{1}{\sigma_n - B}t^\alpha\frac{1}{\sigma_{n+1}-B}\right). \end{aligned} \tag{6.9}$$

We now compute the algebra of the Schwinger-Dyson operators, as in section 4, by changing the contour $C$ to the imaginary axis. For the commutators of the operators with the same superscript, the result is found to coincide with that of the corresponding matrix model operators, after making the identification:

$$g_{\mathrm{st}} \leftrightarrow \frac{1}{N}, \quad y_i \leftrightarrow \zeta_i, \quad x_i \leftrightarrow \sigma_i.$$

They are

$$[\partial_y T_A^A(y), \partial_{y'}T_A^A(y')] = -g_{\mathrm{st}}^2\partial_y\partial_{y'}(\partial_y - \partial_{y'})\frac{1}{y-y'}(T_A^A(y) - T_A^A(y')), \tag{6.10}$$

$$\begin{aligned} [\partial_y T_A^A(y), T_1^A(y';y',x')] &= -g_{\mathrm{st}}^2\partial_y\left\{\frac{-2}{(y-y')^2}(T_1^A(y;y',x') + T_1^A(y';y,x'))\right. \\ &+ \left.\frac{1}{(y-y')^2}(T_1^A(y;y,x') + 3T_1^A(y';y',x')) + \frac{1}{y-y'}\partial_{y'}T_1^A(y';y',x')\right\}, \end{aligned} \tag{6.11}$$

$\cdots.$

Similar result is obtained for the operators with the superscript $B$.

For commutators between the operators with different superscripts, the situation is not so simple, except for the trivial case

$$[\partial_y T^A_A(y), \partial_x T^B_B(x)] = 0. \tag{6.12}$$

For example, for the commutator $[\partial_y T^A_A(y), T^B_1(y', x'; x')]$, the closed algebra

$$[\partial_y T^A_A(y), T^B_1(y', x'; x')] = g^2_{\rm st}\partial_y \partial_{y'} \frac{1}{y-y'}(T^B_1(y, x'; x') - T^B_1(y', x'; x')), \tag{6.13}$$

which coincides with the result of the matrix model operators, can be obtained only when the formula

$$\partial_y \int_C \frac{dx}{2\pi i} D_z(x, x') D_w(x, x') \frac{\delta}{\delta \tilde{J}_2(y, z, y', w)} = 0 \tag{6.14}$$

holds. To check this formula, we need to know a more detailed property of the functional derivatives $\frac{\delta}{\delta \tilde{J}_n}$, $(n \geq 2)$.

Here we try to justify this formula by using the short-domain expansion of the functional derivative operators. It will be useful also for the derivation of the $W_3$ constraints discussed in the next section. The short-domain expansions, in general, depend on the choice of a background. Since a specific background was already assumed in taking the continuum limit of the two-matrix model, use of the short-domain expansion should be allowed here, as we have already done in the one-matrix cases in studying the closure of the Schwinger-Dyson algebra as a consistency check of the continuum results. The difference between the present case and the one-matrix cases is that the tadpole term is absorbed into the definition of the spin-flip amplitudes. Because of this peculiarity, the Hamiltonian (5.34) itself apparently contains no terms which fix the dimensions of the loop operators. In order to fix the dimensions, however, it is sufficient to give the dimension of the Hamiltonian, since it then determines the dimensions of the loop operators and the string coupling $g_{\rm st}$ uniquely. Following our result of a direct derivation of the Hamiltonian from the two-matrix model, we assume the dimension of $\mathcal{H}$ to be $[\mathcal{H}] = [y]^{1/3}$. Also, it is noted that a new dimensional parameter can enter as the integration constant of the Schwinger-Dyson equation $\partial_y T^A_A(y) Z[J] = 0$ or $\partial_x T^B_B(x) Z[J] = 0$. We can relate this parameter to the cosmological constant with dimension two ($[T] = [y]^2$), characterizing the $c = 1/2$ background. See the next section, in particular, eq. (7.20), for more details.

Let us now write down the short-domain expansions. Firstly, for the operators with a single domain, $\frac{\delta}{\delta \tilde{J}_A}$, $\frac{\delta}{\delta \tilde{J}_B}$, by considering the dimensions of the loop operators and the cosmological constant, we can assume the expansions analogous to the one-matrix case (B.20)

$$\frac{\delta}{\delta \tilde{J}_A(y)} = a^A_{4/3} y^{4/3} + a^A_{-2/3} T y^{-2/3} + \sum_u A_u y^{-u-1} \quad (y : \text{large}), \tag{6.15}$$

$$\frac{\delta}{\delta \tilde{J}_B(x)} = a^B_{4/3} x^{4/3} + a^B_{-2/3} T x^{-2/3} + \sum_u B_u x^{-u-1} \quad (x : \text{large}), \tag{6.16}$$

where $a$'s stand for dimensionless constants, and $u$ runs over positive integers divided by 3. The first two terms come from the disk singular terms (thus, $a^A_{4/3} = a^B_{4/3}$, $a^A_{-2/3} = a^B_{-2/3}$), and the remaining terms represent both the contributions of the large $y$ or $x$ expansion of the cylinder singular parts and those of the local operators.

Secondly, referring to the explicit form of $w^{(2)}(y,x)$ in (C.27), we can assume that the operator $\frac{\delta}{\delta \tilde{J}_1(y,x)}$ has the following large $x$ expansion:

$$\begin{aligned}
\frac{\delta}{\delta \tilde{J}_1(y,x)} \quad = \quad & a^1_{5/3} x^{5/3} + a^1_{2/3} x^{2/3} y + x^{-1/3}(a^1_{-1/3} y^2 + a'^1_{-1/3} T) \\
& + x^{-4/3}(a^1_{-4/3} y^3 + a'^1_{-4/3} T y + a''^1_{-4/3} B_{2/3}) \\
& + x^{-7/3}(a^1_{-7/3} y^4 + a'^1_{-7/3} T y^2 + a''^1_{-7/3} B_{2/3} y + a'''^1_{-7/3} B_{5/3}) \\
& + a^1_{1/3} x^{1/3} \frac{\delta}{\delta \tilde{J}_A(y)} + a^1_{-2/3} x^{-2/3} y \frac{\delta}{\delta \tilde{J}_A(y)} \\
& + x^{-5/3}((a^1_{-5/3} y^2 + a'^1_{-5/3} T) \frac{\delta}{\delta \tilde{J}_A(y)} + a''^1_{-5/3} B_1) \\
& + x^{-7/3} a^{(\mathrm{iv})1}_{-7/3} B_{1/3} \frac{\delta}{\delta \tilde{J}_A(y)} \\
& + x^{-1}(a^1_{-1} \mathcal{O}_1(\frac{\delta}{\delta \tilde{J}})(y) + a'^1_{-1} B_{1/3}) \\
& + x^{-2}(a^1_{-2} y \mathcal{O}_1(\frac{\delta}{\delta \tilde{J}})(y) + a'^1_{-2} B_{1/3} y + a''^1_{-2} B_{4/3}) \\
& + x^{-7/3} a^{(\mathrm{v})1}_{-7/3} \mathcal{O}_2(\frac{\delta}{\delta \tilde{J}})(y) + O(x^{-8/3}),
\end{aligned} \tag{6.17}$$

where $\mathcal{O}_m(\frac{\delta}{\delta \tilde{J}})(y)$ represents the loop operator with a microscopic $m$-spin flipped domain being added to the domain $y$, and the remaining notations are the same as before. Here, we assume that the nonpolynomial powers of $y$ all come from the loop operators with the macroscopic $y$ domain, and that the fractional powers of $T$ appear only through the $B$'s. For every disk amplitude whose explicit form is derived in the Appendix C, we can confirm these properties. The above form (6.17) is then a consequence of the dimensional analysis. Note that we do not have to worry about the ordering of $B$'s and loop operators, because $[B_u, \frac{\delta}{\delta \tilde{J}_I}] = 0$ due to $[\frac{\delta}{\delta \tilde{J}_B}, \frac{\delta}{\delta \tilde{J}_I}] = 0$.

Similarly, the large $x$ expansions of $\frac{\delta}{\delta \tilde{J}_2(y,x,y',x')}$, $\frac{\delta}{\delta \tilde{J}_2(y,x,y',x)}$ are assumed to be, respectively,

$$\begin{aligned}
\frac{\delta}{\delta \tilde{J}_2(y,x,y',x')} \quad = \quad & a^2_1 x + a^2_{2/3} x^{2/3} D_z(y,y') \frac{\delta}{\delta \tilde{J}_A(z)} \\
& + a^2_{1/3} x^{1/3} D_z(y,y') \frac{\delta}{\delta \tilde{J}_1(z,x')} \\
& + a^2_0 y + a'^2_0 y' + a''^2_0 x' + O(x^{-1/3}),
\end{aligned} \tag{6.18}$$

$$\frac{\delta}{\delta \tilde{J}_2(y,x,y',x)} \quad = \quad \tilde{a}^2_1 x + \tilde{a}^2_{2/3} x^{2/3} D_z(y,y') \frac{\delta}{\delta \tilde{J}_A(z)}$$

$$+\tilde{a}_0^2 y + \tilde{a}_0'^2 y' + O(x^{-1/3}). \qquad (6.19)$$

We can fix the normalizations of the spin flipped loop operators, referring to the disk results in the Appendix D. For example, $a^1_{-1} = s$. This implies the following relations

$$\begin{aligned}
\mathcal{O}_1(\frac{\delta}{\delta \tilde{J}})(y) &= s^{-1}\int_C \frac{dx}{2\pi i}\frac{\delta}{\delta \tilde{J}_1(y,x)} - s^{-1}a'^1_{-1}B_{1/3}, \\
\mathcal{O}_2(\frac{\delta}{\delta \tilde{J}})(y) &= s^{-1}\int_{C''}\frac{dx'}{2\pi i}\int_{C'}\frac{dy'}{2\pi i}s^{-1}\int_C \frac{dx}{2\pi i}\frac{\delta}{\delta \tilde{J}_2(y,x,y',x')} + c^{-1}s^{-1}\tilde{a}\frac{\delta}{\delta \tilde{J}_A(y)}B_{1/3}, \\
&\cdots,
\end{aligned} \qquad (6.20)$$

with $\tilde{a}$ being some constant, which are a consequence of the above expansions. The first equation of (6.20) is the integrated version of eq. (5.23). Roughly speaking, $\mathcal{O}_m(\frac{\delta}{\delta \tilde{J}})(y)$ is obtained from $\frac{\delta}{\delta \tilde{J}_m}$ by integrating the domains except $y$, and the factor $s^{-1}$ is regarded as a sort of finite renormalization accompanied with flipping of a single spin.

Now substituting the expansions (6.18) and (6.19), and using the formula such as (B.23), we can directly check the validity of the formula (6.14), and thus justify the algebra (6.13). From these results, it is expected that the algebra of higher operators also coincides with those of the matrix model operators before taking the double-scaling limit. In order to prove this for the general case, we need to obtain more precise information with respect to the coefficients $a$'s in the expansions of generic loop operators. We would like to emphasize again that, since we have started from the matrix model which is already an integral solution of the constraints, the closure of the Schwinger-Dyson algebra is guaranteed in our approach. The confirmation of the closure of the algebra is useful, however, as a consistency check of the continuum limit. Since the structure of the algebra of the Schwinger-Dyson operators is essentially determined by the splitting and merging processes of the loops which are closely related with those in the Hamiltonian, it is natural to suppose that the algebra is not affected by the scaling limit, in view of our discussions in section 5.

It is straightforward to see that the matrix model operators (6.9) form a closed algebra such as

$$\begin{gathered}
[T_A^{(\text{mat.})\,A}, T_A^{(\text{mat.})\,A}] \sim T_A^{(\text{mat.})\,A}, \quad [T_B^{(\text{mat.})\,B}, T_B^{(\text{mat.})\,B}] \sim T_B^{(\text{mat.})\,B}, \\
[T_A^{(\text{mat.})\,A}, T_n^{(\text{mat.})\,A}] \sim T_n^{(\text{mat.})\,A}, \quad [T_B^{(\text{mat.})\,B}, T_n^{(\text{mat.})\,B}] \sim T_n^{(\text{mat.})\,B}, \\
[T_n^{(\text{mat.})\,A}, T_m^{(\text{mat.})\,A}] \sim T_{n+m}^{(\text{mat.})\,A}, \quad [T_n^{(\text{mat.})\,B}, T_m^{(\text{mat.})\,B}] \sim T_{n+m}^{(\text{mat.})\,B}, \\
[T_A^{(\text{mat.})\,A}, T_n^{(\text{mat.})\,B}] \sim T_n^{(\text{mat.})\,B}, \quad [T_B^{(\text{mat.})\,B}, T_n^{(\text{mat.})\,A}] \sim T_n^{(\text{mat.})\,A}, \\
[T_A^{(\text{mat.})\,A}, T_B^{(\text{mat.})\,B}] = 0, \quad [T_n^{(\text{mat.})\,A}, T_m^{(\text{mat.})\,B}] \sim T_{n+m}^{(\text{mat.})\,A} + T_{n+m}^{(\text{mat.})\,B}, \\
(n, m \geq 1).
\end{gathered} \qquad (6.21)$$

The corresponding algebra of the continuum generators

$$\partial_y T_A^A(y),\ T_n^A(y; y_1, x_1, \cdots, y_n, x_n),\ \partial_x T_B^B(x),\ T_n^B(y_1, x_1, \cdots, y_n, x_n; x) \quad (n = 1, 2, \cdots) \quad (6.22)$$

must have the same structure. Half of the generators with the superscript $A(B)$ form a subalgebra of the full algebra .** We note that, as already emphasized in the beginning of this section, the generators contained in the Hamiltonian $\mathcal{H}$ are only the particular symmetrized combinations of the general Schwinger-Dyson operators. To ensure the closure of the full algebra, we are lead to introduce all of the above general Schwinger-Dyson operators.

## 7. Reduction to the $W_3$ Constraints

In the previous section, we obtained the huge algebra of the Schwinger-Dyson operators. Although this algebra has a very complicated structure, we will next demonstrate how the $W_3$ constraints, characterizing the $c = 1/2$ noncritical string, is naturally derived from the integrability condition of the first few constraint operators appearing in our Hamiltonian. This will provide us yet another consistency check of the preceding results.

The constraints associated with the Hamiltonian (6.1) are

$$
\begin{aligned}
-\partial_y T_A^A(y) Z[J] &= 0, && (7.1)\\
-\partial_x T_B^B(x) Z[J] &= 0, && (7.2)\\
T_1(y,x) Z[J] &= 0, && (7.3)\\
T_2(y_1,x_1,y_2,x_2) Z[J] &= 0, && (7.4)\\
&\cdots .
\end{aligned}
$$

Considering the closure of this constraint algebra

$$
\begin{aligned}
&[\partial_y T_A^A(y), T_1(y',x)]\\
&= -g_{\mathrm{st}}^2 \partial_y \left\{ \frac{1}{(y-y')} \partial_{y'} T_1(y',x) + \frac{1}{(y-y')^2} (T_1(y,x) + 3T_1(y',x)) \right.\\
&\quad \left. + \frac{-2}{(y-y')^2} (T_1^A(y;y',x) + T_1^A(y';y,x) + T_1^B(y,x;x) + T_1^B(y',x;x)) \right\}, \qquad (7.5)
\end{aligned}
$$

which follows from (6.11) and (6.13), we obtain a new combination of the Schwinger-Dyson operators,

$$
(T_1^A(y;y',x) + T_1^A(y';y,x) + T_1^B(y,x;x) + T_1^B(y',x;x)) Z[J] = 0. \qquad (7.6)
$$

Similarly, from the algebra $[\partial_x T_B^B(x), T_1(y,x')]$, we have

$$
(T_1^B(y,x;x') + T_1^B(y,x';x) + T_1^A(y;y,x) + T_1^A(y;y,x')) Z[J] = 0. \qquad (7.7)
$$

** The algebra similar to this subalgebra is presented in ref. [9].

Let us consider the integrated versions of these:

$$\Big[s^{-1}\int_{C''}\frac{dx}{2\pi i}\int_{C'}\frac{dy'}{2\pi i}(T_1^A(y;y',x)+T_1^A(y';y,x))$$
$$+s^{-1}\int_{C''}\frac{dx}{2\pi i}\int_{C'}\frac{dy'}{2\pi i}T_1^B(y',x;x)\Big]Z[J]=0, \tag{7.8}$$

$$s^{-1}\int_{C''}\frac{dx'}{2\pi i}\int_{C'}\frac{dx}{2\pi i}(T_1^B(y,x;x')+T_1^B(y,x';x))Z[J]=0, \tag{7.9}$$

where the integral contours $C, C'$ and $C''$ successively wrap around the negative real axis and the singularities in the left half plane.

In order to derive the $W_3$ constraints, it is important to examine the explicit forms of these integrated operators. First, by using the formulas in the Appendix B, we have

$$\int_C\frac{dy}{2\pi i}D_z(y,y')\frac{\delta}{\delta\tilde J_I(z,x,y_1,x_1,\cdots)}$$
$$= -\frac{\delta}{\delta\tilde J_I(y',x,y_1,x_1,\cdots)}+\sum_{n=0}^{\infty}y'^n c_n^I(x,y_1,x_1,\cdots), \tag{7.10}$$

where the second term stands for the polynomial part with respect to $y'$ of the large $y'$ expansion of $\frac{\delta}{\delta\tilde J_I(y',x',y_1,x_1,\cdots)}$. The meaning of eq. (7.10) is as follows. In terms of the domain length $l'$ conjugate to $y'$, the l.h.s. represents the operator with the domain length $l'+\varepsilon$, where $\varepsilon$ comes from the procedure of changing the contour $\int_C\frac{dy}{2\pi i}\cdots=\lim_{\varepsilon\to+0}\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\mathrm{e}^{\varepsilon y}\cdots$. When $l'\neq 0$, the limit $\varepsilon\to+0$ is smooth, and thus it coincides with the first term of the r.h.s. However, for the singular terms supported only at $l'=0$, if any, the l.h.s. gives no contribution, since $\lim_{\varepsilon\to+0}\delta^{(n)}(\varepsilon)=0$ $(n=0,1,2,\cdots)$ in the prescription of the Appendix B. The second term is needed to subtract such a contribution. As is seen from the expansions (6.15) $\sim$(6.18), for example, $c_n^I$'s are all zero for $I=A,B,1$, and $c_1^2=a_1^2$, $c_0^2=a_0^2x+a_0'^2x_1+a_0''^2y_1$, the others vanish for $I=2$. Also, for composite operators, the above formula can be used. For instance,

$$\int_C\frac{dy}{2\pi i}D_z(y,y')\frac{\delta}{\delta\tilde J_A(z)}\frac{\delta}{\delta\tilde J_1(z,x)}=-\frac{\delta}{\delta\tilde J_A(y')}\frac{\delta}{\delta\tilde J_1(y',x)}+\sum_{n=0}^{\infty}y'^n c_n^{A1}(x). \tag{7.11}$$

From the short-domain expansions, $c_n^{A1}(x)$'s turn out to be polynomials of $x$ and to vanish for $n\geq 4$. Thus, we have

$$\int_{C''}\frac{dx}{2\pi i}\int_{C'}\frac{dy}{2\pi i}D_z(y,y')\frac{\delta}{\delta\tilde J_A(z)}\frac{\delta}{\delta\tilde J_1(z,x)}=-\frac{\delta}{\delta\tilde J_A(y')}\int_{C''}\frac{dx}{2\pi i}\frac{\delta}{\delta\tilde J_1(y',x)}, \tag{7.12}$$

and similarly

$$\int_{C''}\frac{dx'}{2\pi i}\int_{C'}\frac{dx}{2\pi i}D_z(x,x')\frac{\delta}{\delta\tilde J_B(z)}\frac{\delta}{\delta\tilde J_1(y,z)}=-\int_{C''}\frac{dx'}{2\pi i}\frac{\delta}{\delta\tilde J_B(x')}\frac{\delta}{\delta\tilde J_1(y,x')}. \tag{7.13}$$

After using the above formulas and doing the integral in the r.h.s. of (7.13) by substituting the expanded forms (6.16), (6.17), the integrated operators are written as

$$\begin{aligned}
& s^{-1}\int_{C''}\frac{dx}{2\pi i}\int_{C'}\frac{dy'}{2\pi i}T_1^A(y;y',x) \\
= & -\frac{\delta}{\delta\tilde{J}_A(y)}\mathcal{O}_1(\frac{\delta}{\delta\tilde{J}})(y)-c\mathcal{O}_2(\frac{\delta}{\delta\tilde{J}})(y) \\
& +s^{-1}(\tilde{a}-a'^1_{-1})\frac{\delta}{\delta\tilde{J}_A(y)}B_{1/3} \\
& -g_{\mathrm{st}}^2\int_{-i\infty}^{i\infty}\frac{dy'}{2\pi i}\tilde{J}_A(y')\partial_{y'}D_z(y,y')\mathcal{O}_1(\frac{\delta}{\delta\tilde{J}})(z) \\
& +(\text{terms containing } \tilde{J}_I \quad (I\neq A)),
\end{aligned} \tag{7.14}$$

$$\begin{aligned}
& s^{-1}\int_{C''}\frac{dx'}{2\pi i}\int_{C'}\frac{dx}{2\pi i}T_1^B(y,x;x') \\
= & -s^{-1}a^B_{4/3}a^1_{-7/3}y^4-s^{-1}(a^B_{4/3}a'^1_{-7/3}+a^B_{-2/3}a^1_{-1/3})Ty^2 \\
& -s^{-1}(a^B_{4/3}a''^1_{-7/3}+a^1_{2/3})B_{2/3}y \\
& -s^{-1}(a^B_{4/3}a'''^1_{-7/3}+a^1_{5/3})B_{5/3}-s^{-1}a^B_{-2/3}a'^1_{-1/3}T^2 \\
& -c(1+c^{-1}s^{-1}a^B_{4/3}a^{(\mathrm{v})1}_{-7/3})\mathcal{O}_2(\frac{\delta}{\delta\tilde{J}})(y) \\
& +s^{-1}(\tilde{a}-a^B_{4/3}a^{(\mathrm{iv})1}_{-7/3}-a^1_{1/3})\frac{\delta}{\delta\tilde{J}_A(y)}B_{1/3} \\
& +(\text{terms containing } \tilde{J}_I \quad (I\neq A)).
\end{aligned} \tag{7.15}$$

The disk parts of (7.14) and (7.15) can be interpreted to represent the continuum versions of the Schwinger-Dyson equations (C.2) and (C.3), respectively. In (7.15), thanks to the short-domain expansions, the integral $\int_C\frac{dx'}{2\pi i}\frac{\delta}{\delta\tilde{J}_B(x')}\frac{\delta}{\delta\tilde{J}_1(y,x')}$ whose $y$-dependence was difficult to see in this form turns out to be essentially a polynomial of $y$. This property is crucial for the derivation of the $W_3$ constraints.

Also, from the results (D.12) and (D.13) in the Appendix D, we can assume the following identity holds:

$$\begin{aligned}
& s^{-1}\int_{C''}\frac{dx'}{2\pi i}\int_{C'}\frac{dy'}{2\pi i}s^{-1}\int_C\frac{dx}{2\pi i}\frac{\delta}{\delta\tilde{J}_2(y,x,y',x')} \\
= & \ s^{-1}\int_{C''}\frac{dx'}{2\pi i}\int_{C'}\frac{dy'}{2\pi i}s^{-1}\int_C\frac{dx}{2\pi i}\frac{\delta}{\delta\tilde{J}_2(y',x,y,x')} \\
= & \ s^{-1}\int_{C''}\frac{dx'}{2\pi i}\int_{C'}\frac{dx}{2\pi i}s^{-1}\int_C\frac{dy'}{2\pi i}\frac{\delta}{\delta\tilde{J}_2(y,x,y',x')} \\
= & \ s^{-1}\int_{C''}\frac{dx'}{2\pi i}\int_{C'}\frac{dx}{2\pi i}s^{-1}\int_C\frac{dy'}{2\pi i}\frac{\delta}{\delta\tilde{J}_2(y,x',y',x)}.
\end{aligned} \tag{7.16}$$

In the Appendix D, we have confirmed these equations for some simple cases. This means that the results of the spin-flip processes are independent of their orderings. By using these identities for the explicit forms of $T_1^A$, $T_1^B$, we can see the symmetry properties:

$$
\begin{aligned}
& s^{-1}\int_{C''}\frac{dx}{2\pi i}\int_{C'}\frac{dy'}{2\pi i}T_1^A(y;y',x)Z[J]\Big|_{J_I=0\ (I\neq A)} \\
= \;& s^{-1}\int_{C''}\frac{dx}{2\pi i}\int_{C'}\frac{dy'}{2\pi i}T_1^A(y';y,x)Z[J]\Big|_{J_I=0\ (I\neq A)},
\end{aligned}
$$

$$
\begin{aligned}
& s^{-1}\int_{C''}\frac{dx'}{2\pi i}\int_{C'}\frac{dx}{2\pi i}T_1^B(y,x;x')Z[J]\Big|_{J_I=0\ (I\neq A)} \\
= \;& s^{-1}\int_{C''}\frac{dx'}{2\pi i}\int_{C'}\frac{dx}{2\pi i}T_1^B(y,x';x)Z[J]\Big|_{J_I=0\ (I\neq A)}.
\end{aligned}
\tag{7.17}
$$

Setting $\tilde{J}_I = 0$ $(I \neq A)$, eqs. (7.8) and (7.9) become respectively

$$
\begin{aligned}
& [-\frac{\delta}{\delta\tilde{J}_A(y)}\mathcal{O}_1(\frac{\delta}{\delta\tilde{J}})(y) + \mathcal{O}'_0 - c\mathcal{O}_2(\frac{\delta}{\delta\tilde{J}})(y) + s^{-1}(\tilde{a} - a'^1_{-1})\frac{\delta}{\delta\tilde{J}_A(y)}B_{1/3} \\
- \;& g_{\rm st}^2\int_{-i\infty}^{i\infty}\frac{dy'}{2\pi i}\tilde{J}_A(y')\partial_{y'}D_z(y,y')\mathcal{O}_1(\frac{\delta}{\delta\tilde{J}})(z)]Z[J]\Big|_{J_I=0\ (I\neq A)} = 0,
\end{aligned}
\tag{7.18}
$$

$$
\begin{aligned}
& [-s^{-1}a^B_{4/3}a^1_{-7/3}y^4 - s^{-1}(a^B_{4/3}a'^1_{-7/3} + a^B_{-2/3}a^1_{-1/3})Ty^2 \\
- \;& s^{-1}(a^B_{4/3}a''^1_{-7/3} + a^1_{2/3})B_{2/3}y - s^{-1}(a^B_{4/3}a'''^1_{-7/3} + a^1_{5/3})B_{5/3} - s^{-1}a^B_{-2/3}a'^1_{-1/3}T^2 \\
- \;& c(1 + c^{-1}s^{-1}a^B_{4/3}a^{({\rm v})1}_{-7/3})\mathcal{O}_2(\frac{\delta}{\delta\tilde{J}})(y) \\
+ \;& s^{-1}(\tilde{a} - a^B_{4/3}a^{({\rm iv})1}_{-7/3} - a^1_{1/3})\frac{\delta}{\delta\tilde{J}_A(y)}B_{1/3}]Z[J]\Big|_{J_I=0\ (I\neq A)} = 0.
\end{aligned}
\tag{7.19}
$$

We used (7.14), (7.15) and (7.17), and $\mathcal{O}'_0$ is the $y$-independent operator

$$
\mathcal{O}'_0 = \frac{1}{2}s^{-1}\int_{C''}\frac{dx}{2\pi i}\int_{C'}\frac{dy'}{2\pi i}T_1^B(y',x;x).
$$

Also, from the once-integrated version of (7.1), we have

$$
\begin{aligned}
& [-\frac{\delta^2}{\delta\tilde{J}_A(y)^2} - c\mathcal{O}_1(\frac{\delta}{\delta\tilde{J}})(y) + \mathcal{O}''_0 \\
- \;& g_{\rm st}^2\int_{-i\infty}^{i\infty}\frac{dy'}{2\pi i}\tilde{J}_A(y')\partial_{y'}D_z(y,y')\frac{\delta}{\delta\tilde{J}_A(z)}]Z[J]\Big|_{J_I=0\ (I\neq A)} = 0,
\end{aligned}
\tag{7.20}
$$

where $\mathcal{O}_0''$ is a $y$-independent operator introduced as an integration constant.

The three eqs. (7.18), (7.19) and (7.20) lead to a closed equation of the loop operator $\tilde{\Phi}_A(y)$:

$$
\begin{aligned}
&[a_4 y^4 + a_2 T y^2 + a_1 B_{2/3} y + a_0 B_{5/3} + a_0' T^2 + c\mathcal{O}_0' \\
&\quad + \frac{\delta^3}{\delta \tilde{J}_A(y)^3} - \frac{\delta}{\delta \tilde{J}_A(y)}(\mathcal{O}_0'' + a_A B_{1/3}) + \frac{1}{2} g_{\mathrm{st}}^2 \partial_y^2 \frac{\delta}{\delta \tilde{J}_A(y)} \\
&\quad + g_{\mathrm{st}}^2 \int_{-i\infty}^{i\infty} \frac{dy'}{2\pi i} \tilde{J}_A(y') \partial_{y'} D_z(y, y') \left( \frac{\delta}{\delta \tilde{J}_A(z)} \frac{\delta}{\delta \tilde{J}_A(y)} + \frac{\delta^2}{\delta \tilde{J}_A(z)^2} \right) \\
&\quad + g_{\mathrm{st}}^4 \int_{-i\infty}^{i\infty} \frac{dy'}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dy''}{2\pi i} \tilde{J}_A(y') \tilde{J}_A(y'') \partial_{y'} \partial_{y''} D_z(y, y'') D_w(z, y') \frac{\delta}{\delta \tilde{J}_A(w)}] \\
&\times \quad Z[J]|_{J_I=0\ (I\neq A)} = 0.
\end{aligned}
\tag{7.21}
$$

The coefficients $a$'s are defined by

$$
a_4 = \frac{cs^{-1} a^B_{4/3} a^1_{-7/3}}{1 + c^{-1}s^{-1} a^B_{4/3} a^{(\mathrm{v})1}_{-7/3}}, \quad a_2 = \frac{cs^{-1}(a^B_{4/3} a'^1_{-7/3} + a^B_{-2/3} a^1_{-1/3})}{1 + c^{-1}s^{-1} a^B_{4/3} a^{(\mathrm{v})1}_{-7/3}},
$$

$$
a_1 = \frac{cs^{-1}(a^B_{4/3} a''^1_{-7/3} + a^1_{2/3})}{1 + c^{-1}s^{-1} a^B_{4/3} a^{(\mathrm{v})1}_{-7/3}}, \quad a_0 = \frac{cs^{-1}(a^B_{4/3} a'''^1_{-7/3} + a^1_{5/3})}{1 + c^{-1}s^{-1} a^B_{4/3} a^{(\mathrm{v})1}_{-7/3}},
$$

$$
a_0' = \frac{cs^{-1} a^B_{-2/3} a'^1_{-1/3}}{1 + c^{-1}s^{-1} a^B_{4/3} a^{(\mathrm{v})1}_{-7/3}}, \quad a_A = \frac{cs^{-1}(\tilde{a} - a^B_{4/3} a^{(\mathrm{iv})1}_{-7/3} - a^1_{1/3})}{1 + c^{-1}s^{-1} a^B_{4/3} a^{(\mathrm{v})1}_{-7/3}} - cs^{-1}(\tilde{a} - a'^1_{-1}).
$$

Further, by rescaling as

$$
\frac{\delta}{\delta \tilde{J}_A(y)} \to \left( \frac{-a_4}{16} \right)^{1/3} \frac{\delta}{\delta \tilde{J}_A(y)}, \quad \tilde{J}_A(y) \to \left( \frac{-a_4}{16} \right)^{-1/3} \tilde{J}_A(y),
$$

$$
g_{\mathrm{st}} \to \left( \frac{-a_4}{16} \right)^{1/3} g_{\mathrm{st}}, \quad T \to -\frac{a_4}{a_2} T,
$$

and putting

$$
a_1 B_{2/3} = \frac{a_4}{16} \mathcal{O}_\Delta, \quad a_0 B_{5/3} + a_0' T^2 + c\mathcal{O}_0' = \frac{a_4}{16} \mathcal{O}_1, \quad \mathcal{O}_0'' + a_A B_{1/3} = \left( \frac{-a_4}{16} \right)^{2/3} \mathcal{O}_0,
$$

eq. (7.21) takes the form

$$
\begin{aligned}
&[-16 y^4 + 16 T y^2 - y\mathcal{O}_\Delta - \mathcal{O}_1 \\
&\quad + \frac{\delta^3}{\delta \tilde{J}_A(y)^3} - \frac{\delta}{\delta \tilde{J}_A(y)} \mathcal{O}_0 + \frac{1}{2} g_{\mathrm{st}}^2 \partial_y^2 \frac{\delta}{\delta \tilde{J}_A(y)} \\
&\quad + g_{\mathrm{st}}^2 \int_{-i\infty}^{i\infty} \frac{dy'}{2\pi i} \tilde{J}_A(y') \partial_{y'} D_z(y, y') \left( \frac{\delta}{\delta \tilde{J}_A(z)} \frac{\delta}{\delta \tilde{J}_A(y)} + \frac{\delta^2}{\delta \tilde{J}_A(z)^2} \right)
\end{aligned}
$$

$$
+g_{\rm st}^4\int_{-i\infty}^{i\infty}\frac{dy'}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dy''}{2\pi i}\tilde{J}_A(y')\tilde{J}_A(y'')\partial_{y'}\partial_{y''}D_z(y,y'')D_w(z,y')\frac{\delta}{\delta\tilde{J}_A(w)}]
$$
$$
\times\quad Z[J]\Big|_{J_I=0\ (I\neq A)}=0. \tag{7.22}
$$

Let us now confirm that this equation represents the $W_3$ constraints. In order to do so, we have to identify the disk and cylinder singular parts. For the disk amplitude $w(y)$, the Schwinger-Dyson equation obtained from (7.22) is

$$
w(y)^3-\langle\mathcal{O}_0\rangle_0 w(y)-16y^4+16Ty^2-y\langle\mathcal{O}_\Delta\rangle_0-\langle\mathcal{O}_1\rangle_0=0, \tag{7.23}
$$

where we consider the solution regular except on the negative real axis. Note that this condition does not determine the solution uniquely. In fact, we find two solutions:
i) $\quad w(y)=2^{4/3}(y-\sqrt{\frac{T}{6}})(y+3\sqrt{\frac{T}{6}})^{1/3}$,

$$
\langle\mathcal{O}_0\rangle_0=0,\quad \langle\mathcal{O}_\Delta\rangle_0=128\left(\frac{T}{6}\right)^{3/2},\quad \langle\mathcal{O}_1\rangle_0=-\frac{4}{3}T^2, \tag{7.24}
$$

ii) $\quad w(y)=(y+\sqrt{y^2-T})^{4/3}+(y-\sqrt{y^2-T})^{4/3}$,

$$
\langle\mathcal{O}_0\rangle_0=3T^{4/3},\quad \langle\mathcal{O}_\Delta\rangle_0=0,\quad \langle\mathcal{O}_1\rangle_0=2T^2. \tag{7.25}
$$

It is the solution ii) that reproduces the matrix model result. By repeating the argument in the Appendix C for deriving (C.14), **without** using the $Z_2$ symmetry $A\leftrightarrow B$, we can see that $\langle\mathcal{O}_\Delta\rangle_0$ is proportional to the next leading order ($O(a^3)$) of the universal part of the quantity $\frac{1}{N}\langle{\rm tr}(A-B)\rangle_0$. This implies that the solution i) spontaneously breaks the $Z_2$ symmetry. Here we only consider the $Z_2$ symmetric solution ii) and leave the case i) as a future problem.

For the cylinder amplitude $w(y,y_1)$, the Schwinger-Dyson equation is derived from the lowest order of $g_{\rm st}$ in the $\tilde{J}_A(y_1)$ derivative of (7.22):

$$
\begin{aligned}
&-y\langle\mathcal{O}_\Delta\tilde{\Phi}_A(y_1)\rangle_0-\langle\mathcal{O}_1\tilde{\Phi}_A(y_1)\rangle_0-\langle\mathcal{O}_0\tilde{\Phi}_A(y_1)\rangle_0 w(y)\\
&\quad+\quad(3w(y)^2-\langle\mathcal{O}_0\rangle_0)w(y,y_1)\\
&\quad+\quad g_{\rm st}^2\partial_{y_1}\frac{1}{y-y_1}(2w(y)^2-w(y)w(y_1)-w(y_1)^2)=0.
\end{aligned} \tag{7.26}
$$

If we take the solution ii) as the disk amplitude, note that it satisfies

$$
3w(y)^2-\langle\mathcal{O}_0\rangle_0=0\qquad\text{at } y=0,\pm\sqrt{\frac{T}{2}}. \tag{7.27}
$$

The disk amplitude is regular except the region $y\leq-\sqrt{T}$ on the real axis. By assuming the same property for the cylinder amplitude, eq. (7.26) can be solved, by using the similar

argument as for the case (B.7). The result is

$$
\begin{aligned}
w(y,y_1) &= g_{\rm st}^2 \frac{4}{9}\frac{1}{f(y,y_1)g(y,y_1)}\frac{(y+\sqrt{y^2-T})^{1/3}}{(y+\sqrt{y^2-T})^{2/3}+(y-\sqrt{y^2-T})^{2/3}+T^{1/3}} \\
&\quad \times \frac{(y_1+\sqrt{y_1^2-T})^{1/3}}{(y_1+\sqrt{y_1^2-T})^{2/3}+(y_1-\sqrt{y_1^2-T})^{2/3}+T^{1/3}} \\
&\quad \times \left[1+\frac{3T^{1/3}}{f(y,y_1)}+\frac{3(y+\sqrt{y^2-T})^{1/3}(y_1+\sqrt{y_1^2-T})^{1/3}}{g(y,y_1)}\right], \qquad (7.28)
\end{aligned}
$$

where

$$
\begin{aligned}
f(y,y_1) &= (y+\sqrt{y^2-T})^{1/3}(y_1+\sqrt{y_1^2-T})^{1/3} \\
&\quad +(y-\sqrt{y^2-T})^{1/3}(y_1-\sqrt{y_1^2-T})^{1/3}+T^{1/3}, \\
g(y,y_1) &= (y+\sqrt{y^2-T})^{2/3}+(y_1+\sqrt{y_1^2-T})^{2/3} \\
&\quad +(y+\sqrt{y^2-T})^{1/3}(y_1+\sqrt{y_1^2-T})^{1/3}.
\end{aligned}
$$

From (7.25) and (7.28), the singular parts can be found as

$$
\begin{aligned}
w^{\rm sing}(y) &= 2^{4/3}(y^{4/3}-\frac{T}{3}y^{-2/3}), \qquad (7.29) \\
w^{\rm sing}(y,y_1) &= g_{\rm st}^2\frac{1}{9}\frac{1}{(y-y_1)^2} \\
&\quad \times[y^{2/3}y_1^{-2/3}+2y^{1/3}y_1^{-1/3}-6+2y^{-1/3}y_1^{1/3}+y^{-2/3}y_1^{2/3}]. \qquad (7.30)
\end{aligned}
$$

The connected correlation functions are expanded by the local operator insertions $g_{\alpha_1,\cdots,\alpha_n}$ as

$$
\begin{aligned}
\langle\tilde{\Phi}_A(y_1)\rangle &= w^{\rm sing}(y_1)+g^{(1)}(y_1), \\
\langle\tilde{\Phi}_A(y_1)\tilde{\Phi}_A(y_2)\rangle &= w^{\rm sing}(y_1,y_2)+g^{(2)}(y_1,y_2), \\
\langle\tilde{\Phi}_A(y_1)\cdots\tilde{\Phi}_A(y_K)\rangle &= g^{(K)}(y_1,\cdots,y_K) \quad (K\geq 3), \\
g^{(n)}(y_1,\cdots,y_n) &= \sum_{\alpha_1,\cdots,\alpha_n} g_{\alpha_1,\cdots,\alpha_n} y_1^{-\alpha_1-1}\cdots y_n^{-\alpha_n-1}, \qquad (7.31)
\end{aligned}
$$

where $\alpha_i$'s run over the positive integers $+1/3$ and $+2/3$, i.e. $1/3, 2/3, 4/3, 5/3, 7/3, \cdots$.

Using (7.29)∼(7.31), we expand (7.22) similarly as in the argument of the Virasoro constraints in the Appendix B. From here the analysis is parallel to the reference [19], where the $W_3$ constraints were explicitly derived from the two-matrix model for the first time. So, we show only the results. For $\mathcal{O}_0$, $\mathcal{O}_\Delta$, $\mathcal{O}_1$ insertions,

$$
\langle\mathcal{O}_0\rangle = 3\cdot 2^{4/3}g_{1/3},
$$

$$
\begin{aligned}
\langle \mathcal{O}_0 \tilde{\Phi}_A(y_1) \rangle &= g_{\rm st}^2 2^{4/3} \frac{1}{3} y_1^{-2/3} + 3 \cdot 2^{4/3} \sum_{\alpha_1} g_{1/3,\alpha_1} y_1^{-\alpha_1 - 1}, \\
\langle \mathcal{O}_0 \tilde{\Phi}_A(y_1) \cdots \tilde{\Phi}_A(y_K) \rangle &= 3 \cdot 2^{4/3} \sum_{\alpha_1, \cdots, \alpha_K} g_{1/3,\alpha_1,\cdots,\alpha_K} y_1^{-\alpha_1 - 1} \cdots y_K^{-\alpha_K - 1}, \\
\langle \mathcal{O}_\Delta \rangle &= 3 \cdot 2^{8/3} g_{2/3}, \\
\langle \mathcal{O}_\Delta \tilde{\Phi}_A(y_1) \rangle &= g_{\rm st}^2 2^{8/3} \frac{2}{3} y_1^{-1/3} + 3 \cdot 2^{8/3} \sum_{\alpha_1} g_{2/3,\alpha_1} y_1^{-\alpha_1 - 1}, \\
\langle \mathcal{O}_\Delta \tilde{\Phi}_A(y_1) \cdots \tilde{\Phi}_A(y_K) \rangle &= 3 \cdot 2^{8/3} \sum_{\alpha_1, \cdots, \alpha_K} g_{2/3,\alpha_1,\cdots,\alpha_K} y_1^{-\alpha_1 - 1} \cdots y_K^{-\alpha_K - 1}, \\
\langle \mathcal{O}_1 \rangle &= \frac{16}{3} T^2 + 3 \cdot 2^{8/3} g_{5/3}, \\
\langle \mathcal{O}_1 \tilde{\Phi}_A(y_1) \rangle &= g_{\rm st}^2 2^{8/3} \frac{5}{3} y_1^{2/3} + 3 \cdot 2^{8/3} \sum_{\alpha_1} g_{5/3,\alpha_1} y_1^{-\alpha_1 - 1}, \\
\langle \mathcal{O}_1 \tilde{\Phi}_A(y_1) \cdots \tilde{\Phi}_A(y_K) \rangle &= 3 \cdot 2^{8/3} \sum_{\alpha_1, \cdots, \alpha_K} g_{5/3,\alpha_1,\cdots,\alpha_K} y_1^{-\alpha_1 - 1} \cdots y_K^{-\alpha_K - 1} \\
& \qquad\qquad\qquad\qquad (K \geq 2).
\end{aligned} \tag{7.32}
$$

This means that $\mathcal{O}_0$, $\mathcal{O}_\Delta$, $\mathcal{O}_1$ are expressed in terms of the loop operator $\frac{\delta}{\delta \tilde{J}_A}$

$$
\begin{aligned}
\mathcal{O}_0 &= 3 \cdot 2^{4/3} \int_C \frac{dy}{2\pi i} y^{1/3} \frac{\delta}{\delta \tilde{J}_A(y)}, \\
\mathcal{O}_\Delta &= 3 \cdot 2^{8/3} \int_C \frac{dy}{2\pi i} y^{2/3} \frac{\delta}{\delta \tilde{J}_A(y)}, \\
\mathcal{O}_1 &= \frac{16}{3} T^2 + 3 \cdot 2^{8/3} \int_C \frac{dy}{2\pi i} y^{5/3} \frac{\delta}{\delta \tilde{J}_A(y)},
\end{aligned}
$$

as operators acting on $Z[J]|_{J_I=0 \ (I \neq A)}$.

For the other contributions, by introducing the generating function

$$
\ln Z(\mu) = \sum_{n_\alpha = 0}^{\infty} \frac{\mu_{1/3}^{n_{1/3}}}{n_{1/3}!} \frac{\mu_{2/3}^{n_{2/3}}}{n_{2/3}!} \cdots g_{\underbrace{1/3,\cdots,1/3}_{n_{1/3}},\underbrace{2/3,\cdots,2/3}_{n_{2/3}},\cdots}, \tag{7.33}
$$

they can be expressed as

$$
\begin{aligned}
L_n Z(\mu) = 0 & \qquad (n \geq -1), \\
W_{n'} Z(\mu) = 0 & \qquad (n' \geq -2),
\end{aligned} \tag{7.34}
$$

where

$$
L_{-1} = 2 \cdot 2^{4/3} \frac{\partial}{\partial \mu_{4/3}} + g_{\rm st}^2 \frac{2}{3} \sum_\alpha \alpha \mu_\alpha \frac{\partial}{\partial \mu_{\alpha - 1}} + g_{\rm st}^4 \frac{4}{81} \mu_{2/3} (\mu_{1/3} - g_{\rm st}^{-2} 3 \cdot 2^{4/3} T),
$$

$$
\begin{aligned}
L_0 &= 2\cdot 2^{4/3}\left(\frac{\partial}{\partial\mu_{7/3}}-\frac{T}{3}\frac{\partial}{\partial\mu_{1/3}}\right)+g_{\rm st}^2\frac{2}{3}\sum_\alpha \alpha\mu_\alpha\frac{\partial}{\partial\mu_\alpha}+g_{\rm st}^2\frac{2}{27},\\
L_l &= 2\cdot 2^{4/3}\left(\frac{\partial}{\partial\mu_{l+7/3}}-\frac{T}{3}\frac{\partial}{\partial\mu_{l+1/3}}\right)+\sum_{\beta+\beta'=l}\frac{\partial^2}{\partial\mu_\beta\partial\mu_{\beta'}}+g_{\rm st}^2\frac{2}{3}\sum_\alpha \alpha\mu_\alpha\frac{\partial}{\partial\mu_{\alpha+l}}\\
&\qquad (l\geq 1),
\end{aligned}
\tag{7.35}
$$

$$
\begin{aligned}
W_{-2} &= 3\cdot 2^{8/3}\left(\frac{\partial}{\partial\mu_{8/3}}-\frac{2}{3}T\frac{\partial}{\partial\mu_{2/3}}\right)\\
&\quad +g_{\rm st}^2 2\cdot 2^{4/3}\sum_\alpha \alpha\mu_\alpha\left(\frac{\partial}{\partial\mu_{\alpha+1/3}}-\frac{T}{3}\frac{\partial}{\partial\mu_{\alpha-5/3}}\right)\\
&\quad +g_{\rm st}^2\sum_\alpha \alpha\mu_\alpha\sum_{\beta+\beta'=\alpha-2}\frac{\partial^2}{\partial\mu_\beta\partial\mu_{\beta'}}\\
&\quad +g_{\rm st}^4\frac{1}{3}\sum_{\alpha,\alpha'}\alpha\alpha'\mu_\alpha\mu_{\alpha'}\frac{\partial}{\partial\mu_{\alpha+\alpha'-2}}\\
&\quad +g_{\rm st}^6\frac{4}{3^5}\mu_{4/3}(\mu_{1/3}-g_{\rm st}^{-2}3\cdot 2^{4/3}T)^2+g_{\rm st}^6\frac{8}{3^6}\mu_{2/3}^3,\\
W_{-1} &= 3\cdot 2^{8/3}\left(\frac{\partial}{\partial\mu_{11/3}}-\frac{2}{3}T\frac{\partial}{\partial\mu_{5/3}}\right)+3\cdot 2^{4/3}\frac{\partial^2}{\partial\mu_{2/3}^2}\\
&\quad +g_{\rm st}^2 2\cdot 2^{4/3}\sum_\alpha \alpha\mu_\alpha\left(\frac{\partial}{\partial\mu_{\alpha+4/3}}-\frac{T}{3}\frac{\partial}{\partial\mu_{\alpha-2/3}}\right)\\
&\quad +g_{\rm st}^2\sum_\alpha \alpha\mu_\alpha\sum_{\beta+\beta'=\alpha-1}\frac{\partial^2}{\partial\mu_\beta\partial\mu_{\beta'}}\\
&\quad +g_{\rm st}^4\frac{1}{3}\sum_{\alpha,\alpha'}\alpha\alpha'\mu_\alpha\mu_{\alpha'}\frac{\partial}{\partial\mu_{\alpha+\alpha'-1}}\\
&\quad +g_{\rm st}^6\frac{1}{3^6}(\mu_{1/3}-g_{\rm st}^{-2}3\cdot 2^{4/3}T)^3,\\
W_m &= 3\cdot 2^{8/3}\left(\frac{\partial}{\partial\mu_{m+14/3}}-\frac{2}{3}T\frac{\partial}{\partial\mu_{m+8/3}}+\frac{T^2}{9}\frac{\partial}{\partial\mu_{m+2/3}}\right)\\
&\quad +3\cdot 2^{4/3}\left(\sum_{\beta+\beta'=m+7/3}-\frac{T}{3}\sum_{\beta+\beta'=m+1/3}\right)\frac{\partial^2}{\partial\mu_\beta\partial\mu_{\beta'}}\\
&\quad +\sum_{\beta+\beta'+\beta''=m}\frac{\partial^3}{\partial\mu_\beta\partial\mu_{\beta'}\partial\mu_{\beta''}}\\
&\quad +g_{\rm st}^2 2\cdot 2^{4/3}\sum_\alpha \alpha\mu_\alpha\left(\frac{\partial}{\partial\mu_{\alpha+m+7/3}}-\frac{T}{3}\frac{\partial}{\partial\mu_{\alpha+m+1/3}}\right)\\
&\quad +g_{\rm st}^2\sum_\alpha \alpha\mu_\alpha\sum_{\beta+\beta'=\alpha+m}\frac{\partial^2}{\partial\mu_\beta\partial\mu_{\beta'}}
\end{aligned}
$$

$$+g_{\rm st}^4 \frac{1}{3}\sum_{\alpha,\alpha'} \alpha\alpha' \mu_\alpha \mu_{\alpha'} \frac{\partial}{\partial \mu_{\alpha+\alpha'+m}} \qquad (m \geq 0). \tag{7.36}$$

This is nothing but the $W_3$ constraints.

## 8. Conclusion

Let us first summarize what we have done. We have started our paper by discussing the nature of PSFTs from the view point of the stochastic quantization of the matrix models. Then, we have presented detailed derivations of the stochastic Hamiltonians in the double-scaling limit from the matrix model, and investigated the infinite algebras of the Schwinger-Dyson operators appearing in the Hamiltonians. We have also checked that the algebras contain the Virasoro (one-matrix model) and $W_3$ algebras (two-matrix model), as they should. Proofs of some of the crucial formulas have not been completed, because of technical complexity. It is therefore desirable to develop more powerful methods of treating the double-scaling limit for general target spaces.

After these calculations, we have to reconsider the questions raised in the earlier sections of the present paper. Perhaps, one of the most important lessons of our work is that the structure of the general splitting and merging interactions of string fields with arbitrary matter configuration is not affected by the mixing of the string-field components in taking the scaling limit which is defined for a specific background. This seems to imply that the structure of these terms is completely independent of the backgrounds. Recalling the general discussion in section 2, we realize that the purely cubic Hamiltonian of the matrix model with most general source terms and no bare action already captures the structure of the continuum Hamiltonian in a background-independent way. This conforms to earlier suggestions [21] and points to an intriguing possibility of formulating a background-independent string field theory, encompassing critical strings, by starting from general matrix integrals with infinite number of different matrices. For the case $c \leq 1$, a related idea has already been discussed in ref. [15].

Before pursuing such possibilities, there remains, however, many important issues to be solved. Asides the problems mentioned in the Introduction, what is needed to make further progress is a deeper understanding on how to extract real space-time picture of the string theory from matrix models, since matrix models apparently miss some important characteristics [22, 23] of the string dynamics.

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## Appendix

### A $k=3$ critical point and the disk amplitude

Let us begin from deriving the non-even critical potential for $k=3$. It is sufficient to recall the well known formulas in the method of orthogonal polynomials. For the coefficients $S$ and $R$ in the recursion equation for the orthogonal polynomials $P_n(\lambda)=\lambda^n+\cdots$,

$$\lambda P_n(\lambda)=P_{n+1}(\lambda)+S_nP_n(\lambda)+R_nP_{n-1}(\lambda),$$

we have a set of equations [24] in terms of the potential $V$,

$$\begin{aligned} 0 &= \oint_0 \frac{dz}{2\pi i}\frac{1}{z}V'(z+S+\frac{R}{z}), && \text{(A.1)} \\ x &= \oint_0 \frac{dz}{2\pi i}\frac{N}{\beta}V'(z+S+\frac{R}{z}), && \text{(A.2)} \end{aligned}$$

in the sphere limit $N\sim\beta\to\infty$, where $x=\frac{n}{\beta}$, $S=S(x)\simeq S_n$, $R=R(x)\simeq R_n$. Eqs. (A.1) and (A.2) give the relation that implicitly determines the function $R(x)$, of the form

$$x=W(R).$$

Since the free energy is determined by $R$ as

$$\ln Z\sim\sum_{n=1}^{N-1}(N-n)\ln R_n, \tag{A.3}$$

the following behavior of the $W(R)$,

$$W(R)=1-\text{const.}(1-R)^3 \tag{A.4}$$

as $x\to 1$, $R\to 1$, leads to the $k=3$ criticality of the free energy

$$\ln Z\sim(1-\frac{N}{\beta})^{7/3}.$$

This shows that the minimal order of the $k=3$ critical potential is four. After lengthy calculations, we find that the potential

$$V(M)=\frac{\beta}{N}(\frac{g_2}{2}M^2+\frac{g_3}{3}M^3+\frac{1}{20}M^4) \tag{A.5}$$

with (3.19) and (3.20) satisfies all of the above conditions.

Next let us derive the disk amplitude in the sphere approximation. By using the method of ref. [11], the disk amplitude in the large $N$ limit is given by

$$\langle\Phi(\zeta)\rangle_0 = \frac{1}{2}V'(\zeta) + \frac{1}{2}\frac{\beta}{N}(-\frac{1}{5}\zeta^2 + A\zeta + B)\sqrt{(\zeta - b_+)(\zeta - b_-)}, \tag{A.6}$$

where

$$\begin{aligned} A &= -g_3 - \frac{1}{10}(b_+ + b_-), \\ B &= -g_2 - \frac{g_3}{2}(b_+ + b_-) - \frac{1}{20}(b_+ + b_-)^2 - \frac{1}{40}(b_+ - b_-)^2. \end{aligned}$$

By introducing the variable

$$z = 10g_3 + 3(b_+ + b_-), \tag{A.7}$$

the end points of the cut $b_- < \zeta < b_+$ are determined by the equations:

$$-\frac{1}{4}z^4 + 36z^2 - 256z - 720 + 81920z^{-2} = 3888\frac{N}{\beta}, \tag{A.8}$$

$$(b_+ - b_-)^2 = -\frac{2}{27}z^2 + \frac{80}{9} - \frac{2560}{27}z^{-1}. \tag{A.9}$$

The disk amplitude with a microscopic loop can be read off from the coefficient of $\zeta^{-2}$ in the large $\zeta$ expansion of eq. (A.6),

$$\begin{aligned} \left\langle\frac{1}{N}\mathrm{tr}M\right\rangle_0 &= \frac{1}{32}\frac{\beta}{N}(b_+ - b_-)^2[g_2(b_+ + b_-) + g_3(b_+ + b_-)^2 \\ &+ \frac{g_3}{8}(b_+ - b_-)^2 + \frac{3}{20}(b_+ + b_-)^3 + \frac{3}{40}(b_+ + b_-)(b_+ - b_-)^2]. \end{aligned} \tag{A.10}$$

In the scaling limit

$$\frac{N}{\beta} = 1 - a^3t \tag{A.11}$$

($a$ means a lattice spacing), eq. (A.8) is iteratively solved as

$$\begin{aligned} z &= -8\left(1 + a\frac{3}{4}t^{1/3} + a^2\frac{7}{64}t^{2/3} - a^3\frac{71}{1024}t \right. \\ &\left. -a^4\frac{8515}{442368}t^{4/3} + O(a^5)\right). \end{aligned} \tag{A.12}$$

Then $b_+$ and $b_-$ are

$$\begin{aligned} b_+ &= \frac{-5g_3 + 2}{3} - a2t^{1/3} - a^2\frac{7}{24}t^{2/3} - a^3\frac{49}{384}t \\ &-a^4\frac{1205}{165888}t^{4/3} + O(a^5), \end{aligned} \tag{A.13}$$

$$b_- = \frac{-5g_3 - 10}{3} + a^3\frac{5}{16}t + a^4\frac{15}{256}t^{4/3} + O(a^5). \tag{A.14}$$

Also, it can be seen that $\zeta$ should be tuned to the critical value at $b_+$

$$\zeta = \zeta_*(1+ay), \quad \zeta_* = \frac{-5g_3+2}{3}. \tag{A.15}$$

Substituting (A.11)∼(A.15) into (A.6) and (A.10), we have

$$\begin{aligned} \langle \Phi(\zeta) \rangle_0 &= \frac{1}{2}V'(\zeta) + a^{5/2}w(y) + O(a^{7/2}), & \text{(A.16)} \\ \left\langle \frac{1}{N}\mathrm{tr}M \right\rangle_0 &= -\frac{32+25g_3}{15} - a^3\frac{4}{5}t + a^4\frac{3}{4}t^{4/3} + O(a^5), & \text{(A.17)} \end{aligned}$$

where

$$w(y) = -\frac{1}{5}\zeta_*^{5/2}(y^2 - \frac{1}{2}T^{1/3}y + \frac{3}{8}T^{2/3})\sqrt{y+T^{1/3}} \tag{A.18}$$

with $T = (2\zeta_*^{-1})^3 t$.

# B Expression for the local operator $\mathcal{O}_0$

We give the proof of eqs. (4.12) and (4.13) using the Schwinger-Dyson equations. We first rewrite the once-integrated Schwinger-Dyson equation

$$T(y)Z[J] = 0 \tag{B.1}$$

into the relations among the local operators. Here,

$$T(y) = T_0(y) + \int_0^y dy' \tilde{\rho}(y') + \mathcal{O}_1, \tag{B.2}$$

the $y$-independent operator $\mathcal{O}_1$ appears as an integration constant.

Now to correctly extract the local operators, we must identify and subtract the singular parts in the correlation functions

$$\left. \frac{\delta^K \ln Z[J]}{\delta \tilde{J}(y_1) \cdots \delta \tilde{J}(y_K)} \right|_{J=0} = \langle \tilde{\Phi}(y_1) \cdots \tilde{\Phi}(y_K) \rangle. \tag{B.3}$$

It appears only in the disk and cylinder amplitudes. For the disk, from the large $y$ expansion of (3.37)

$$w(y) = y^{5/2} + \frac{5}{16}Ty^{-1/2} - \frac{15}{128}T^{4/3}y^{-3/2} + \cdots, \tag{B.4}$$

we see that the first two terms of (B.4) correspond to the singular part

$$w^{\mathrm{sing}}(y) = y^{5/2} + \frac{5}{16}Ty^{-1/2}, \tag{B.5}$$

because it is known that the disk one-point function of the local operator behaves as $T^{4/3+\Delta}$ ($\Delta \geq 0$) from the analysis of the continuum theory [16].

For the cylinder amplitude, we start with a derivation of the amplitude $w(y, y_1)$ from the lowest order of $g_{\rm st}$ in the Schwinger-Dyson equation

$$\frac{\delta}{\delta \tilde{J}(y_1)} T(y) Z[J] \bigg|_{J=0} = 0, \tag{B.6}$$

that is,

$$2w(y)w(y,y_1) + g_{\rm st}^2 \partial_{y_1} D_z(y,y_1) w(z) - y\langle \mathcal{O}_0 \tilde{\Phi}(y_1)\rangle_0 - \langle \mathcal{O}_1 \tilde{\Phi}(y_1)\rangle_0 = 0. \tag{B.7}$$

It can be easily solved by noting the fact that $w(y)$ given in (3.37) has single zeroes at $y = \alpha, \bar{\alpha}$, where $\alpha, \bar{\alpha}$ are the solutions of the quadratic equation

$$y^2 - \frac{1}{2}T^{1/3} y + \frac{3}{8} T^{2/3} = 0,$$

explicitly

$$\left.\begin{matrix} \alpha \\ \bar{\alpha} \end{matrix}\right\} = \frac{1 \pm i\sqrt{5}}{4} T^{1/3}. \tag{B.8}$$

By setting $y = \alpha$ or $\bar{\alpha}$ in (B.7), the first term vanishes, [††] then $\langle \mathcal{O}_0 \tilde{\Phi}(y_1)\rangle_0$, $\langle \mathcal{O}_1 \tilde{\Phi}(y_1)\rangle_0$ are determined as

$$\begin{aligned} \langle \mathcal{O}_0 \tilde{\Phi}(y_1)\rangle_0 &= g_{\rm st}^2 \partial_{y_1} \sqrt{y_1 + T^{1/3}}, && \text{(B.9)} \\ \langle \mathcal{O}_1 \tilde{\Phi}(y_1)\rangle_0 &= g_{\rm st}^2 \partial_{y_1} (y_1 - \frac{1}{2}T^{1/3}) \sqrt{y_1 + T^{1/3}}. && \text{(B.10)} \end{aligned}$$

Substituting these into (B.7), we obtain

$$w(y, y_1) = \frac{g_{\rm st}^2}{4} \frac{1}{\sqrt{y + T^{1/3}}\sqrt{y_1 + T^{1/3}}} \frac{1}{(\sqrt{y + T^{1/3}} + \sqrt{y_1 + T^{1/3}})^2}. \tag{B.11}$$

Since the cylinder amplitude of the local operators behaves as $T^{1/3+\Delta_1+\Delta_2}$ ($\Delta_1, \Delta_2 \geq 0$), the singular part is identified to be

$$w^{\rm sing}(y, y_1) = g_{\rm st}^2 \frac{1}{4} \frac{1}{(y - y_1)^2} \left( \sqrt{\frac{y}{y_1}} - 2 + \sqrt{\frac{y_1}{y}} \right). \tag{B.12}$$

Thus, the connected correlation functions are expressed as

$$\begin{aligned} \langle \tilde{\Phi}(y_1)\rangle &= w^{\rm sing}(y_1) + g^{(1)}(y_1), \\ \langle \tilde{\Phi}(y_1)\tilde{\Phi}(y_2)\rangle &= w^{\rm sing}(y_1, y_2) + g^{(2)}(y_1, y_2), \\ \langle \tilde{\Phi}(y_1)\cdots\tilde{\Phi}(y_K)\rangle &= g^{(K)}(y_1, \cdots, y_K) \quad (K \geq 3), && \text{(B.13)} \end{aligned}$$

[††] Here, we assumed that $w(y, y_1)$ does not have any poles at $y = \alpha, \bar{\alpha}$. This is justified, since $\tilde{\Phi}(y)$ is regular except in the negative real axis as is seen from its definition.

where $g^{(n)}(y_1,\cdots,y_n)$ is the part interpreted as local operator insertions, and it is expanded by the correlators among local operators $g_{\alpha_1,\cdots,\alpha_n}$

$$g^{(n)}(y_1,\cdots,y_n)=\sum_{\alpha_1,\cdots,\alpha_n} g_{\alpha_1,\cdots,\alpha_n} y_1^{-\alpha_1-1}\cdots y_n^{-\alpha_n-1}, \tag{B.14}$$

where $\alpha_i$'s run over the positive half odd integers $1/2, 3/2, 5/2, \cdots$. Using (B.5), (B.12)∼(B.14), we expand the Schwinger-Dyson equations

$$\frac{\delta^K}{\delta\tilde{J}(y_1)\cdots\delta\tilde{J}(y_K)} T(y) Z[J]\bigg|_{J=0}=0 \quad (K=0,1,2,\cdots), \tag{B.15}$$

and perform the similar analysis as in ref. [14]. For example, the result for $K=0$ is

$$\begin{aligned}
&-y\langle\mathcal{O}_0\rangle-\langle\mathcal{O}_1\rangle+\left(\frac{5}{16}T\right)^2 y^{-1}+g_{\mathrm{st}}^2\frac{1}{16}y^{-2}\\
&\quad+2\sum_\alpha g_\alpha\left(y^{-\alpha+3/2}+\frac{5}{16}Ty^{-\alpha-3/2}\right)+\sum_{\alpha,\alpha'}(g_\alpha g_{\alpha'}+g_{\alpha,\alpha'})y^{-\alpha-\alpha'-2}=0.
\end{aligned}$$

Here, we use the Greek indices $\alpha,\alpha',\beta,\beta'$ for the positive half odd integers. From the first two powers of the large $y$, $\langle\mathcal{O}_0\rangle$, $\langle\mathcal{O}_1\rangle$ are determined as

$$\langle\mathcal{O}_0\rangle=2g_{1/2}, \quad \langle\mathcal{O}_1\rangle=2g_{3/2}.$$

Performing similar analysis for $K\geq 1$ in (B.15), we obtain for $\mathcal{O}_0$, $\mathcal{O}_1$ insertions

$$\begin{aligned}
\langle\mathcal{O}_0\rangle &= 2g_{1/2},\\
\langle\mathcal{O}_0\tilde{\Phi}(y_1)\rangle &= g_{\mathrm{st}}^2\frac{1}{2}y_1^{-1/2}+2\sum_{\alpha_1} g_{1/2,\alpha_1}y_1^{-\alpha_1-1},\\
\langle\mathcal{O}_0\tilde{\Phi}(y_1)\cdots\tilde{\Phi}(y_K)\rangle &= 2\sum_{\alpha_1,\cdots,\alpha_K} g_{1/2,\alpha_1,\cdots,\alpha_K}y_1^{-\alpha_1-1}\cdots y_K^{-\alpha_K-1},\\
\langle\mathcal{O}_1\rangle &= 2g_{3/2},\\
\langle\mathcal{O}_1\tilde{\Phi}(y_1)\rangle &= g_{\mathrm{st}}^2\frac{3}{2}y_1^{1/2}+2\sum_{\alpha_1} g_{3/2,\alpha_1}y_1^{-\alpha_1-1},\\
\langle\mathcal{O}_1\tilde{\Phi}(y_1)\cdots\tilde{\Phi}(y_K)\rangle &= 2\sum_{\alpha_1,\cdots,\alpha_K} g_{3/2,\alpha_1,\cdots,\alpha_K}y_1^{-\alpha_1-1}\cdots y_K^{-\alpha_K-1} \quad (K\geq 2).
\end{aligned} \tag{B.16}$$

This shows that $\langle\mathcal{O}_0\Phi(y_1)\rangle$ and $\langle\mathcal{O}_1\Phi(y_1)\rangle$ have the singular parts $g_{\mathrm{st}}^2\frac{1}{2}y_1^{-1/2}$ and $g_{\mathrm{st}}^2\frac{3}{2}y_1^{1/2}$, respectively.

The other powers of $y$ give the Virasoro constraints. For $K=0$, we have

$$\begin{aligned}
&2\left(g_{l+7/2}+\frac{5}{16}Tg_{l+1/2}\right)+\sum_{\beta+\beta'=l}(g_\beta g_{\beta'}+g_{\beta,\beta'})\\
&\quad+g_{\mathrm{st}}^2\frac{1}{16}\delta_{l,0}+\left(\frac{5}{16}T\right)^2\delta_{l,-1}=0 \quad (l=-1,0,1,\cdots),
\end{aligned}$$

where $g$ with negative indices is understood as zero. For general $K$, by introducing the generating function

$$\ln Z(\mu) = \sum_{n_\alpha=0}^{\infty} \frac{\mu_{1/2}^{n_{1/2}}}{n_{1/2}!} \frac{\mu_{3/2}^{n_{3/2}}}{n_{3/2}!} \cdots g_{\underbrace{1/2,\cdots,1/2}_{n_{1/2}},\underbrace{3/2,\cdots,3/2}_{n_{3/2}},\cdots}, \tag{B.17}$$

the relations can be expressed in the form

$$L_n Z(\mu) = 0 \quad (n \geq -1),$$

$$\begin{aligned}
L_{-1} &= 2\frac{\partial}{\partial \mu_{5/2}} + g_{\mathrm{st}}^2 \sum_\alpha \alpha\mu_\alpha \frac{\partial}{\partial \mu_{\alpha-1}} + (\frac{5}{16}T + g_{\mathrm{st}}^2 \frac{1}{4}\mu_{1/2})^2, \\
L_0 &= 2(\frac{\partial}{\partial \mu_{7/2}} + \frac{5}{16}T\frac{\partial}{\partial \mu_{1/2}}) + g_{\mathrm{st}}^2 \sum_\alpha \alpha\mu_\alpha \frac{\partial}{\partial \mu_\alpha} + g_{\mathrm{st}}^2 \frac{1}{16}, \\
L_l &= 2(\frac{\partial}{\partial \mu_{l+7/2}} + \frac{5}{16}T\frac{\partial}{\partial \mu_{l+1/2}}) + \sum_{\beta+\beta'=l} \frac{\partial^2}{\partial \mu_\beta \partial \mu_{\beta'}} \\
&\quad + g_{\mathrm{st}}^2 \sum_\alpha \alpha\mu_\alpha \frac{\partial}{\partial \mu_{\alpha+l}} \quad (l \geq 1).
\end{aligned} \tag{B.18}$$

Note the operation of $\mathcal{O}_0$ ($\mathcal{O}_1$) on $Z(\mu)$ is expressed as the local operator insertion $2\frac{\partial}{\partial \mu_{1/2}}$ ($2\frac{\partial}{\partial \mu_{3/2}}$).

Next, we notice that as an operator acting on $Z(\mu)$, $\frac{\delta}{\delta \tilde{J}(y)}$ is expanded by the local operator $\frac{\partial}{\partial \mu_\alpha}$:

$$\frac{\delta}{\delta \tilde{J}(y)} = \sum_\alpha \frac{\partial}{\partial \mu_\alpha} y^{-\alpha-1} \quad \text{as acting on } Z(\mu).$$

Furthermore, since the partition function $Z[J]$ is related to the $Z(\mu)$ through the rescaling

$$Z[J] = \exp[\int \frac{dy}{2\pi i} \tilde{J}(y) w^{\mathrm{sing}}(y) + \frac{1}{2}\int \frac{dy_1}{2\pi i}\int \frac{dy_2}{2\pi i} \tilde{J}(y_1)\tilde{J}(y_2) w^{\mathrm{sing}}(y_1, y_2)] Z(\mu), \tag{B.19}$$

we see that

$$\frac{\delta}{\delta \tilde{J}(y)} = w^{\mathrm{sing}}(y) + \int \frac{dy_1}{2\pi i} \tilde{J}(y_1) w^{\mathrm{sing}}(y, y_1) + \sum_\alpha \frac{\partial}{\partial \mu_\alpha} y^{-\alpha-1} \tag{B.20}$$

as acting on $Z[J]$.

After these preparations, we can now prove the eqs. (4.12) and (4.13). First, we consider (4.12). The integral along the contour $C$ is defined by the analytic continuation using the Beta function. For example,

$$\begin{aligned}
\int_C \frac{dy}{2\pi i} y^A &= \int_{-\infty}^0 \frac{dy}{2\pi i} (y - i0)^A + \int_0^{-\infty} \frac{dy}{2\pi i} (y + i0)^A \\
&= -\frac{\sin \pi A}{\pi} \int_0^\infty dy\, y^A = -\frac{\sin \pi A}{\pi} B(A+1, -A-1) \\
&= -\frac{1}{A+1}\frac{1}{\Gamma(0)}.
\end{aligned}$$

For $A \neq -1$ this is zero, and for $A = -1$, because the integrand has no cut, the contour can be deformed as encircling the origin

$$\int_C \frac{dy}{2\pi i} y^{-1} = \oint_0 \frac{dy}{2\pi i} y^{-1} = 1.$$

We summarize these into

$$\int_C \frac{dy}{2\pi i} y^A = \delta_{A,-1}. \tag{B.21}$$

From this result, in the case that the pole $y = -y_1$ exists inside the contour, we obtain

$$\begin{aligned}
\int_C \frac{dy}{2\pi i} \frac{y^A}{y + y_1} &= \sum_{n=0}^{\infty} (-y_1)^n \int_C \frac{dy}{2\pi i} y^{A-n-1} \\
&= \left\{ \begin{array}{ll} (-y_1)^A & A = 0, 1, 2, \cdots \\ 0 & \text{otherwise.} \end{array} \right.
\end{aligned} \tag{B.22}$$

Also, when the pole $y = y_1$ is outside the contour,

$$\int_C \frac{dy}{2\pi i} \frac{y^A}{y - y_1} = \left\{ \begin{array}{ll} 0 & A = 0, 1, 2, \cdots \\ -y_1^A & \text{otherwise.} \end{array} \right. \tag{B.23}$$

By using these formulas (B.21) and (B.23), we can see that

$$2 \int_C \frac{dy}{2\pi i} y^{1/2} \sum_\alpha \frac{\partial}{\partial \mu_\alpha} y^{-\alpha-1} = 2 \frac{\partial}{\partial \mu_{1/2}},$$

$$2 \int_C \frac{dy}{2\pi i} y^{1/2} w^{\text{sing}}(y) = 0 = \langle \mathcal{O}_0 \rangle^{\text{sing}},$$

$$2 \int_C \frac{dy}{2\pi i} y^{1/2} w^{\text{sing}}(y, y_1) = g_{\text{st}}^2 \frac{1}{2} y_1^{-1/2} = \langle \mathcal{O}_0 \Phi(y_1) \rangle^{\text{sing}},$$

which just mean that $\mathcal{O}_0$ is written as in (4.12).

Next, we verify the second equality (4.13). For this purpose, it is sufficient to show that

$$\lim_{\varepsilon \to +0} \int_{-i\infty}^{i\infty} \frac{dy}{2\pi i} \mathrm{e}^{\varepsilon y} y^A = \int_C \frac{dy}{2\pi i} y^A \tag{B.24}$$

for arbitrary $A$, because the formulas such as (B.22) and (B.23) can be derived from (B.21).

We consider the following integral in the case $A \notin \boldsymbol{Z}$

$$\begin{aligned}
\int_{-i\infty}^{i\infty} \frac{dy}{2\pi i} y^A &= \frac{1}{\pi} \cos \frac{\pi A}{2} \int_0^\infty dx\, x^A = \frac{1}{\pi} \cos \frac{\pi A}{2} B(A+1, -A-1) \\
&= \frac{1}{\Gamma(0)} \frac{1}{A+1} \frac{1}{2 \sin \frac{\pi A}{2}} = 0.
\end{aligned}$$

This implies that for $A \notin \boldsymbol{Z}$,

$$\lim_{\varepsilon\to+0}\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\mathrm{e}^{\varepsilon y}y^{A} = \lim_{\varepsilon\to+0}\sum_{n=0}^{\infty}\frac{\varepsilon^{n}}{n!}\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}y^{n+A} = 0. \tag{B.25}$$

Also, when $A = -1, -2, -3, \cdots$, the contour can be deformed to the circle enclosing the origin,

$$\begin{aligned}\lim_{\varepsilon\to+0}\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\mathrm{e}^{\varepsilon y}y^{A} &= \lim_{\varepsilon\to+0}\oint_{0}\frac{dy}{2\pi i}\mathrm{e}^{\varepsilon y}y^{A}\\ &= \lim_{\varepsilon\to+0}\frac{\varepsilon^{|A|-1}}{(|A|-1)!} = \delta_{A,-1}.\end{aligned}$$

Further, for $A = 0, 1, 2, \cdots$, the derivative of the $\delta$-function at $y = \varepsilon$ appears. In our prescription, since $\delta^{(A)}(\varepsilon) = 0$ for finite $\varepsilon$, the limit $\varepsilon \to +0$ is also zero

$$\lim_{\varepsilon\to+0}\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\mathrm{e}^{\varepsilon y}y^{A} = \lim_{\varepsilon\to+0}\delta^{(A)}(\varepsilon) = 0.$$

These results are summarized into

$$\lim_{\varepsilon\to+0}\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\mathrm{e}^{\varepsilon y}y^{A} = \delta_{A,-1}. \tag{B.26}$$

Comparing this with (B.21), we conclude that eq. (4.13) holds. Similarly, it is easy to see that the operator $\mathcal{O}_1$ is expressed as

$$\begin{aligned}\mathcal{O}_1 &= 2\int_C\frac{dy}{2\pi i}y^{3/2}\frac{\delta}{\delta\tilde{J}(y)}, &\text{(B.27)}\\ &= 2\lim_{\varepsilon\to+0}\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\mathrm{e}^{\varepsilon y}y^{3/2}\frac{\delta}{\delta\tilde{J}(y)}. &\text{(B.28)}\end{aligned}$$

## C Disk Amplitudes in the Two-Matrix Model

Here, we obtain various disk amplitudes (genus zero one-point functions) in the two-matrix model by using the continuum limit of the Schwinger-Dyson equations which give the relations among them. Some of the disk amplitudes before taking the continuum limit have been obtained by Staudacher [20]. We will extend his results considerably and give detailed forms of the continuum disk amplitudes that have not appeared in the literatures.

We introduce the following notations for the disk amplitudes (some of which are borrowed from [20]):

$$W_n = \left\langle\frac{1}{N}\mathrm{tr}A^n\right\rangle_0,$$

$$
\begin{aligned}
W^{(2)}_{n,m} &= \left\langle \frac{1}{N}\mathrm{tr}A^nB^m \right\rangle_0, \\
W(\zeta) &= \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{\zeta - A} \right\rangle_0, \\
W(\sigma) &= \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{\sigma - B} \right\rangle_0, \\
W_j(\zeta) &= \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{\zeta - A}B^j \right\rangle_0, \\
W^{(2)}(\zeta,\sigma) &= \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{\zeta - A}\frac{1}{\sigma - B} \right\rangle_0, \\
W_j(\zeta_1;\zeta_2,\sigma_2,\cdots,\zeta_k,\sigma_k) &= \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{\zeta_1 - A}B^j\frac{1}{\zeta_2 - A}\frac{1}{\sigma_2 - B}\cdots\frac{1}{\zeta_k - A}\frac{1}{\sigma_k - B} \right\rangle_0, \\
W^{(2k)}(\zeta_1,\sigma_1,\cdots,\zeta_k,\sigma_k) &= \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{\zeta_1 - A}\frac{1}{\sigma_1 - B}\cdots\frac{1}{\zeta_k - A}\frac{1}{\sigma_k - B} \right\rangle_0 \quad (k=1,2,\cdots).
\end{aligned}
$$

### *C.1* $W(\zeta)$

Let us first start from the disk amplitude with the simplest spin configuration on the loop. (The spins on the loop are all $A$-.) It is obtained by combining the following three Schwinger-Dyson equations:

$$
\begin{aligned}
(\zeta - g\zeta^2)W(\zeta) &= cW_1(\zeta) + W(\zeta)^2 + 1 - g(\zeta + W_1), && \text{(C.1)} \\
(\zeta - g\zeta^2)W_1(\zeta) &= cW_2(\zeta) + W(\zeta)W_1(\zeta) + W_1 - g(\zeta W_1 + W^{(2)}_{1,1}), && \text{(C.2)} \\
W_1(\zeta) - gW_2(\zeta) &= c\zeta W(\zeta) - c. && \text{(C.3)}
\end{aligned}
$$

We can eliminate $W_1(\zeta)$ and $W_2(\zeta)$, and have a cubic equation of $W(\zeta)$:

$$
W(\zeta)^3 + a_1W(\zeta)^2 + a_2W(\zeta) + a_3 = 0, \tag{C.4}
$$

$$
\begin{aligned}
a_1 &= \frac{c}{g} - 2(\zeta - g\zeta^2), \\
a_2 &= (\zeta - g\zeta^2)^2 - \frac{c}{g}(\zeta - g\zeta^2) + (\frac{c^3}{g} - g)\zeta + 1 - gW_1, \\
a_3 &= (-1 + gW_1 + g\zeta)(\zeta - g\zeta^2) + (1 - 3c + cg\zeta)W_1 \\
&\quad -g^2W_3 - g + \frac{c}{g}(1 - c^2) - c\zeta,
\end{aligned}
$$

where in order to eliminate $W_2$ and $W^{(2)}_{1,1}$ we used

$$
W_1 - gW_2 = cW_1, \quad W_2 - gW_3 = cW^{(2)}_{1,1} + 1.
$$

The expressions of $W_1$ and $W_3$ are evaluated by the orthogonal polynomial method [17, 18] as follows:

$$\begin{aligned} W_1 &= \frac{1}{64g^3}[3\rho^4 - 6c\rho^3 - 2(1-2c)\rho^2 - 2c(1-2c)^2\rho^{-1} \\ &\quad +32g^2 - (1-2c+4c^2)(1-2c)], \end{aligned} \tag{C.5}$$

$$\begin{aligned} W_3 &= \frac{1}{16\cdot 64g^5}[-16(\rho^6-1) + 90c(\rho^5-1) \\ &\quad + \left(80(1-2c) - \frac{531}{4}c^2\right)(\rho^4-1) \\ &\quad +(-64-94c+380c^2+60c^3)(\rho^3-1) \\ &\quad +(-48+336c-333c^2-150c^3-54c^4)(\rho^2-1) \\ &\quad +2(1-2c)(32-41c-14c^2-66c^3)(\rho-1) \\ &\quad -6c(1-2c)^2(7-14c-2c^2)(\rho^{-1}-1) \\ &\quad +c(1-2c)^2(16-21c-6c^2-6c^3)(\rho^{-2}-1) \\ &\quad -4c^3(1-2c)^3(\rho^{-3}-1) - \frac{3}{4}c^2(1-2c)^4(\rho^{-4}-1)], \end{aligned} \tag{C.6}$$

where $\rho$ is implicitly determined by

$$\begin{aligned} g^2 &= -\frac{1}{32}[4\rho^3 - 9c\rho^2 - 4(1-2c)\rho + 2c(1-2c+2c^2) \\ &\quad -c(1-2c)^2\rho^{-2}]. \end{aligned} \tag{C.7}$$

In the continuum limit, expanding $g$ and $\rho$ about the critical points

$$g_* = \sqrt{10c_*^3}, \quad \rho_* = 3c_* \quad (c_* = \frac{-1+2\sqrt{7}}{27}),$$

eq. (C.7) can be solved iteratively:

$$\begin{aligned} \rho &= \rho_* + a^{2/3}\frac{2}{3}\rho_*(5t)^{1/3} + a^{4/3}\frac{5}{36}\rho_*(5t)^{2/3} \\ &\quad -a^2\frac{35}{288}\rho_* t - a^{8/3}\frac{8557}{311040}\rho_*(5t)^{4/3} - a^{10/3}\frac{3523}{746496}\rho_*(5t)^{5/3} \\ &\quad +a^4\frac{21205}{442368}\rho_* t^2 + O(a^{14/3}) \end{aligned} \tag{C.8}$$

where $g$ is expanded as $g = g_*(1-a^2t)$.

Substituting this into eqs. (C.5) and (C.6), we have $W_1$ and $W_3$ in the expanded form:

$$W_1 = W_1^{\mathrm{non}} + \hat{W}_1, \tag{C.9}$$

$$W_1^{\mathrm{non}} = \frac{-8\rho_*^4 + 3(2g_*)^2}{3(2g_*)^3} + a^2\frac{-136\rho_*^4 + 27(2g_*)^2}{27(2g_*)^3}t,$$

$$\begin{aligned}\hat{W}_1 &= a^{8/3}\frac{8\rho_*^4}{27(2g_*)^3}(5t)^{4/3}+a^{10/3}\frac{4\rho_*^4}{81(2g_*)^3}(5t)^{5/3}\\&\quad+a^4\frac{-8527\rho_*^4+972(2g_*)^2}{972(2g_*)^3}t^2+O(a^{14/3}),\end{aligned}$$

$$\begin{aligned}W_3 &= W_3^{\rm non}+\hat{W}_3, \qquad (C.10)\\ W_3^{\rm non} &= \frac{32(420-839\rho_*)\rho_*^5}{729(2g_*)^5}+a^2\frac{160(252-611\rho_*)\rho_*^5}{729(2g_*)^5}t,\\ \hat{W}_3 &= a^{8/3}\frac{320\rho_*^6}{81(2g_*)^5}(5t)^{4/3}+a^{10/3}\frac{160\rho_*^6}{243(2g_*)^5}(5t)^{5/3}\\&\quad+a^4\frac{70(1152-3593\rho_*)\rho_*^5}{729(2g_*)^5}t^2+O(a^{14/3}),\end{aligned}$$

where we denoted the non-universal pieces by $W_1^{\rm non}, W_3^{\rm non}$ and the universal ones which give the continuum limit by $\hat{W}_1, \hat{W}_3$.

Now, we shall evaluate $W(\zeta)$ in the continuum limit. Shifting $W(\zeta)$ as

$$W(\zeta)=-\frac{a_1}{3}+\hat{W}(\zeta), \qquad (C.11)$$

eq. (C.4) becomes

$$\hat{W}(\zeta)^3-\frac{1}{3}A_2\hat{W}(\zeta)-\frac{1}{27}A_1=0 \qquad (C.12)$$

where

$$A_1=9a_1a_2-2a_1^3-27a_3, \quad A_2=a_1^2-3a_2.$$

Then the critical point of $\zeta$ denoted by $P_*$ is determined by

$$A_1|_*=A_2|_*=0, \qquad (C.13)$$

where $|_*$ means that $g, W_1$ and $W_3$ are set to the critical values. It turns out that eq. (C.13) gives a cubic equation of $P_*$, and its solution is threefold: $P_*=\frac{1+3c_*}{2g_*}$.

After substituting $\zeta=P_*(1+ay)$ into (C.12) and expanding with respect to $a$, (C.12) becomes

$$\hat{W}(\zeta)^3-\frac{a^{8/3}cs^{8/3}}{40\cdot 2^{2/3}}T^{4/3}\hat{W}(\zeta)-\frac{a^4c^{3/2}s^4}{160\sqrt{10}}(16y^4-16Ty^2+2T^2)+O(a^{13/3})=0, \qquad (C.14)$$

where $c$ is fixed to be the critical value $c_*$, $s$ is the irrational number $s=2+\sqrt{7}$, and the rescaled variable $T=\frac{20}{s^2}t$ is introduced.

The solution of (C.14) is

$$\begin{aligned}\hat{W}(\zeta) &= a^{4/3}\frac{c^{1/2}s^{4/3}}{\sqrt{10}\cdot 2^{4/3}}[(y+\sqrt{y^2-T})^{4/3}+(y-\sqrt{y^2-T})^{4/3}]+O(a^{5/3})\\ &\equiv a^{4/3}\frac{c^{1/2}s^{4/3}}{\sqrt{10}\cdot 2^{4/3}}w(y)+O(a^{5/3}), \qquad (C.15)\end{aligned}$$

which gives the universal part of the disk amplitude.

Also, the non-universal part $W^{\rm non}(\zeta)$ is

$$W^{\rm non}(\zeta) = -\frac{a_1}{3} = -\frac{c}{3g} + \frac{2}{3}(\zeta - g\zeta^2). \tag{C.16}$$

*C.2* $W_1(\zeta)$, $W_2(\zeta)$

The amplitude $W_1(\zeta)$ ($W_2(\zeta)$) represents the configuration that the spins on the loop all align $A$- except a small $B$-domain consisting of a single spin (two spins).

From (C.1),

$$\begin{aligned} W_1(\zeta) &= \frac{1}{c}[(\zeta - g\zeta^2)W^{\rm non}(\zeta) - W^{\rm non}(\zeta)^2 - 1 + g(\zeta + W_1^{\rm non})] \\ &+\frac{1}{c}[\zeta - g\zeta^2 - 2W^{\rm non}(\zeta)]\hat{W}(\zeta) \\ &+\frac{1}{c}[-\hat{W}(\zeta)^2 + g\hat{W}_1]. \end{aligned} \tag{C.17}$$

We identify the universal and non-universal parts as follows. If there are polynomials of $y$ and $T$, they are non-universal. Also, if there are amplitudes, with the spin configurations simpler than that of $W_1(\zeta)$, multiplied by polynomials of $y$ and $T$, they are non-universal. After these identifications, the remaining terms are universal. By using this rule, the universal and non-universal parts, denoted by $\hat{W}_1(\zeta)$ and $W_1^{\rm non}(\zeta)$ respectively, are determined as

$$W_1(\zeta) = W_1^{\rm non}(\zeta) + \hat{W}_1(\zeta), \tag{C.18}$$

$$\begin{aligned} W_1^{\rm non}(\zeta) &= \frac{1}{c}[(\zeta - g\zeta^2)W^{\rm non}(\zeta) - W^{\rm non}(\zeta)^2 - 1 + g(\zeta + W_1^{\rm non})] \\ &+\frac{1}{c}[\zeta - g\zeta^2 - 2W^{\rm non}(\zeta)]\hat{W}(\zeta), \end{aligned} \tag{C.19}$$

$$\begin{aligned} \hat{W}_1(\zeta) &= \frac{1}{c}[-\hat{W}(\zeta)^2 + g\hat{W}_1] \\ &= a^{8/3}\frac{s^{8/3}}{40\cdot 2^{2/3}}[-(y+\sqrt{y^2-T})^{8/3} - (y-\sqrt{y^2-T})^{8/3} + T^{4/3}] + O(a^3) \\ &\equiv a^{8/3}\frac{s^{8/3}}{40\cdot 2^{2/3}}w_1(y) + O(a^3). \end{aligned} \tag{C.20}$$

By a similar manipulation for eq. (C.2) using

$$\hat{W}(\zeta)^2 = -c\hat{W}_1(\zeta) + g\hat{W}_1,$$

we have

$$\begin{aligned} W_2(\zeta) &= W_2^{\rm non}(\zeta) + \hat{W}_2(\zeta), \\ W_2^{\rm non}(\zeta) &= \left(\frac{1}{3g} + \frac{1}{3c}(\zeta - g\zeta^2)\right)\Bigg[-\frac{1}{c} + \frac{g}{c}\zeta - \frac{c}{9g^2} + \frac{1}{9g}(\zeta - g\zeta^2) \end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{9c}(\zeta - g\zeta^2)^2 + \frac{g}{c}W_1^{\rm non}\Big] \\
&-\frac{g}{c}\left(\frac{2}{3g} - \frac{1}{3c}(\zeta - g\zeta^2)\right)\hat{W}_1 - \frac{1}{c}(2 - \frac{1}{c} - g\zeta)W_1 - \frac{g^2}{c^2}W_3 - \frac{g}{c^2} \\
&+\left[\frac{1}{3g^2} - \frac{1}{3c^2}(\zeta - g\zeta^2)^2 + \frac{1}{c^2}(1 - g\zeta) - \frac{g}{c^2}W_1^{\rm non}\right]\hat{W}(\zeta) \\
&+\frac{1}{g}\hat{W}_1(\zeta) + O(a^{13/3}),
\end{aligned}
$$

$$
\begin{aligned}
\hat{W}_2(\zeta) &= -\frac{1}{c}\hat{W}(\zeta)\hat{W}_1(\zeta) + O(a^{13/3}) \\
&= a^4\frac{s^4}{160\sqrt{10c}}(16y^4 - 16Ty^2 + 2T^2) + O(a^{13/3}) \\
&\equiv a^4\frac{s^4}{160\sqrt{10c}}w_2(y) + O(a^{13/3}).
\end{aligned}
$$

### *C.3* $W^{(2)}(\zeta,\sigma)$, $W^{(4)}(\zeta_1,\sigma_1,\zeta_2,\sigma_2)$

To discuss the higher disk amplitudes, the recursion equation for $W^{(2k)}$ given by Staudacher (eq. (20) in ref. [20]) is useful:

$$
\begin{aligned}
&W^{(2k)}(P_1, Q_1, \cdots, P_k, Q_k) = \frac{1}{P_1 - gP_1^2 - cQ_1 - W(P_1)} \\
&\times \quad \{D_Q(Q_1, Q_k)\left(\frac{g}{c}[Q - gQ^2 - W(Q_1) - W(Q_k)] - 1 + gP_1\right) \\
&\qquad \times W^{(2k-2)}(P_2, Q_2, \cdots, P_k, Q) \\
&\qquad -\frac{g^2}{c}D_P(P_2, P_k)W^{(2k-4)}(P, Q_2, \cdots, P_{k-1}, Q_{k-1}) \\
&\qquad +\frac{g}{c}\sum_{l=2}^{k-1}[D_Q(Q_1, Q_l)W^{(2l-2)}(P_2, Q_2, \cdots, P_l, Q)] \\
&\qquad\qquad \times [D_Q(Q_l, Q_k)W^{(2k-2l)}(P_{l+1}, Q_{l+1}, \cdots, P_k, Q)] \\
&\qquad +cD_P(P_1, P_2)W^{(2k-2)}(P, Q_2, \cdots, P_k, Q_k) \\
&\qquad -\sum_{l=2}^{k}W^{(2l-2)}(P_l, Q_1, \cdots, P_{l-1}, Q_{l-1}) \\
&\qquad\qquad \times D_P(P_1, P_l)W^{(2k+2-2l)}(P, Q_l, \cdots, P_k, Q_k)\}. \qquad (C.21)
\end{aligned}
$$

Let us consider the continuum limit for the cases $k = 1$ and $k = 2$:

$$
W^{(2)}(\zeta,\sigma) = \frac{(1 - g\zeta)W(\sigma) - cW(\zeta) - gW_1(\sigma)}{\zeta - g\zeta^2 - c\sigma - W(\zeta)}, \qquad (C.22)
$$

$$
W^{(4)}(\zeta_1,\sigma_1,\zeta_2,\sigma_2) = \frac{1}{\zeta_2 - g\zeta_2^2 - c\sigma_2 - W(\zeta_2)}
$$

$$\times\{(c-W^{(2)}(\zeta_1,\sigma_2))D_\zeta(\zeta_1,\zeta_2)W^{(2)}(\zeta,\sigma_1)$$
$$+D_\sigma(\sigma_1,\sigma_2)[\frac{g}{c}(\sigma-g\sigma^2-W(\sigma_1)-W(\sigma_2))-1+g\zeta_2]W^{(2)}(\zeta_1,\sigma)$$
$$+\frac{g^2}{c}W(\zeta_1)\}. \tag{C.23}$$

For $k=1$, putting $\zeta=P_*(1+ay)$, $\sigma=P_*(1+ax)$ and expanding with respect to $a$, (C.22) becomes

$$\begin{aligned}
W^{(2)}(\zeta,\sigma) &= W^{(2)\,\mathrm{non}}(\zeta,\sigma)+\hat{W}^{(2)}(\zeta,\sigma), && \text{(C.24)}\\
W^{(2)\,\mathrm{non}}(\zeta,\sigma) &= c(1-as(y+x))+\sqrt{10c}(\hat{W}(\zeta)+\hat{W}(\sigma)), && \text{(C.25)}\\
\hat{W}^{(2)}(\zeta,\sigma) &= a^{-1}\frac{10}{s}\frac{1}{y+x}(-\hat{W}(\zeta)\hat{W}(\sigma)-\hat{W}(\zeta)^2+c\hat{W}_1(\sigma))\\
&\quad +a^{-2}\frac{10^{3/2}}{c^{1/2}s^2}\frac{\hat{W}(\zeta)^2\hat{W}(\sigma)+\hat{W}(\zeta)\hat{W}(\sigma)^2}{(y+x)^2}\\
&\quad +a^2\frac{cs^2}{120}\frac{1}{(y+x)^2}[160(y^4+x^4)+80(yx^3+y^3x)-40y^2x^2\\
&\qquad +12(s-10)T(y^2+x^2)+24(s-5)Tyx-120T(y^2+x^2+yx)\\
&\qquad +12sT(y+x)^2+15T^2]\\
&\quad +O(a^{7/3})\\
&\equiv a^{5/3}\frac{cs^{5/3}}{4\cdot 2^{2/3}}w^{(2)}(y,x)+O(a^2), && \text{(C.26)}
\end{aligned}$$

where

$$w^{(2)}(y,x)=\frac{-w(y)^2-w(y)w(x)-w(x)^2+3T^{4/3}}{y+x}, \tag{C.27}$$

and we calculated $\hat{W}^{(2)}(\zeta,\sigma)$ up to the next leading order since it will be necessary to obtain $W^{(4)}(\zeta_1,\sigma_1,\zeta_2,\sigma_2)$ below.

Repeating the same procedure for $k=2$, we have

$$W^{(4)}(\zeta_1,\sigma_1,\zeta_2,\sigma_2)=W^{(4)\,\mathrm{non}}(\zeta_1,\sigma_1,\zeta_2,\sigma_2)+\hat{W}^{(4)}(\zeta_1,\sigma_1,\zeta_2,\sigma_2), \tag{C.28}$$

$$\begin{aligned}
W^{(4)\,\mathrm{non}}(\zeta_1,\sigma_1,\zeta_2,\sigma_2) &= 10c^2-10c[D_\zeta(\zeta_1,\zeta_2)\hat{W}(\zeta)+D_\sigma(\sigma_1,\sigma_2)\hat{W}(\sigma)]\\
&\quad -\sqrt{10c}[D_\zeta(\zeta_1,\zeta_2)(\hat{W}^{(2)}(\zeta,\sigma_1)+\hat{W}^{(2)}(\zeta,\sigma_2))\\
&\quad +D_\sigma(\sigma_1,\sigma_2)(\hat{W}^{(2)}(\zeta_1,\sigma)+\hat{W}^{(2)}(\zeta_2,\sigma))], && \text{(C.29)}
\end{aligned}$$

$$\hat{W}^{(4)}(\zeta_1,\sigma_1,\zeta_2,\sigma_2)=a\frac{5c^2s}{8}w^{(4)}(y_1,x_1,y_2,x_2)+O(a^{4/3})$$
$$=\ a\frac{5c^2s}{8}\frac{1}{(y_1-y_2)(x_1-x_2)}\Big[-\frac{8}{3}(y_1-y_2)(x_1-x_2)(y_1+y_2+x_1+x_2)$$

$$
\begin{aligned}
&-\frac{1}{2}(w(y_1)+w(x_1)+2w(y_2)+2w(x_2))w^{(2)}(y_1,x_1)\\
&+\frac{1}{2}(w(y_1)+w(x_2)+2w(y_2)+2w(x_1))w^{(2)}(y_1,x_2)\\
&+\frac{1}{2}(w(y_2)+w(x_1)+2w(y_1)+2w(x_2))w^{(2)}(y_2,x_1)\\
&-\frac{1}{2}(w(y_2)+w(x_2)+2w(y_1)+2w(x_1))w^{(2)}(y_2,x_2)\Big]\\
&+O(a^{4/3}).
\end{aligned}
\tag{C.30}
$$

We note that the above results have the symmetry under the cyclic permutation of variables

$$
\zeta_1 \to \sigma_2, \quad \sigma_2 \to \zeta_2, \quad \zeta_2 \to \sigma_1, \quad \sigma_1 \to \zeta_1,
$$

which should be satisfied by the definition of the amplitude. In general, however, in the expressions such as eq. (C.23) the symmetry is totally obscure. Obtaining amplitudes with correct symmetry property thus constitutes a quite nontrivial consistency check for the continuum results.

*C.4* $W_1(\zeta_1;\zeta_2,\sigma_2)$

From the following Schwinger-Dyson equation

$$
0=\int d^{N^2}A d^{N^2}B \sum_{\alpha=1}^{N^2} \frac{\partial}{\partial A_\alpha}\left(\mathrm{tr}\left(\frac{1}{\zeta_1-A}t^\alpha \frac{1}{\zeta_2-A}\frac{1}{\sigma_2-B}\right)\mathrm{e}^{-S}\right),
$$

we have

$$
\begin{aligned}
W_1(\zeta_1;\zeta_2,\sigma_2) \;=\; & -\frac{1}{c}[D_\zeta(\zeta_1,\zeta_2)(\zeta-g\zeta^2-W(\zeta_1)-W(\zeta_2))W^{(2)}(\zeta,\sigma_2)\\
& +gW(\sigma_2)].
\end{aligned}
\tag{C.31}
$$

By a similar calculation as before, we find

$$
W_1(\zeta_1;\zeta_2,\sigma_2)=W_1^{\mathrm{non}}(\zeta_1;\zeta_2,\sigma_2)+\hat{W}_1(\zeta_1;\zeta_2,\sigma_2),
\tag{C.32}
$$

$$
\begin{aligned}
W^{\mathrm{non}}(\zeta_1;\zeta_2,\sigma_2) &= -D_\zeta(\zeta_1,\zeta_2)(\zeta-g\zeta^2)(1-\frac{s}{P_*}(\zeta+\sigma_2-2P_*))\\
&+\frac{s}{P_*}\left(\frac{2c}{3g}-\frac{2}{3}(\zeta_1-g\zeta_1^2)-\frac{2}{3}(\zeta_2-g\zeta_2^2)\right)+g\left(\frac{1}{3g}-\frac{2}{3c}(\sigma_2-g\sigma_2^2)^2\right)\\
&-\frac{s}{P_*}(\hat{W}(\zeta_1)+\hat{W}(\zeta_2))-\frac{g}{c}\hat{W}(\sigma_2)\\
&-\sqrt{\frac{10}{c}}\left(\frac{2c}{3g}-\frac{2}{3}(\zeta_1-g\zeta_1^2)-\frac{2}{3}(\zeta_2-g\zeta_2^2)\right)D_\zeta(\zeta_1,\zeta_2)\hat{W}(\zeta)
\end{aligned}
$$

$$
\begin{aligned}
&-\sqrt{\frac{10}{c}}D_\zeta(\zeta_1,\zeta_2)((\zeta-g\zeta^2)(\hat{W}(\zeta)+\hat{W}(\sigma_2)))\\
&-\sqrt{10c}D_\zeta(\zeta_1,\zeta_2)\hat{W}_1(\zeta)-\frac{1}{c}D_\zeta(\zeta_1,\zeta_2)((\zeta-g\zeta^2)\hat{W}^{(2)}(\zeta,\sigma_2))\\
&-\frac{1}{c}\left(\frac{2c}{3g}-\frac{2}{3}(\zeta_1-g\zeta_1^2)-\frac{2}{3}(\zeta_2-g\zeta_2^2)\right)D_\zeta(\zeta_1,\zeta_2)\hat{W}^{(2)}(\zeta,\sigma_2),
\end{aligned}
\tag{C.33}
$$

$$
\begin{aligned}
\hat{W}_1(\zeta_1;\zeta_2,\sigma_2) &= \frac{1}{c}(\hat{W}(\zeta_1)+\hat{W}(\zeta_2))D_\zeta(\zeta_1,\zeta_2)\hat{W}^{(2)}(\zeta,\sigma_2)\\
&= a^2\frac{cs^2}{16}(w(y_1)+w(y_2))D_y(y_1,y_2)w^{(2)}(y,x_2)\\
&\quad +O(a^{7/3}).
\end{aligned}
\tag{C.34}
$$

These results show the following scaling behavior for the general disk amplitudes $W^{(2k)}(\zeta_1,\sigma_1,\cdots,\zeta_k,\sigma_k)$:

$$
\hat{W}^{(2k)}(\zeta_1,\sigma_1,\cdots,\zeta_k,\sigma_k)=a^{\frac{7}{3}-\frac{2}{3}k}w^{(2k)}(y_1,x_1,\cdots,y_k,x_k),
\tag{C.35}
$$

which is consistent with the analysis of the boundary conformal field theory [5]. An argument for this is as follows: The gravitationally dressed spin operator exists at the boundary of domains and its dimension is $[y]^{2/3}$. This is derived by considering the gravitational dressing of the spin operator, whose dimension is $[y]^{1/2}$, in the boundary conformal field theory in flat space [25]. In eq. (C.35), increasing $k$ by one unit corresponds to adding the two domains. Clearly, the boundaries of domains are also increased by two. Then, the dimension of $w^{(2k)}$ is changed by a factor

$$
[y]^{2\cdot\frac{2}{3}+2\cdot(-1)}=[y]^{-\frac{2}{3}},
$$

where $2\cdot\frac{2}{3}$ comes from the dressed spin operators at the two boundaries, and $2\cdot(-1)$ from the two domains. This coincides with (C.35).

# D Continuum Spin-Flip Operator

In the matrix model before taking the scaling limit, a domain consisting of only a single flipped spin can be obtained as an integral of a general domain,

$$
\frac{1}{N}\mathrm{tr}\left(\frac{1}{\zeta-A}B\cdots\right)=\oint\frac{d\sigma}{2\pi i}\sigma\frac{1}{N}\mathrm{tr}\left(\frac{1}{\zeta-A}\frac{1}{\sigma-B}\cdots\right).
\tag{D.1}
$$

Let us construct the continuum version of this operation. We can do this by deriving the relation between the universal parts of the both sides in (D.1).

First, let us consider $\hat{W}^{(2)}(\zeta,\sigma)$ and $\hat{W}_1(\zeta)$. Comparing eqs. (C.20) and (C.27), we have

$$
\int_C\frac{dx}{2\pi i}w^{(2)}(y,x)=-w(y)^2+T^{4/3}=w_1(y)-2T^{4/3},
\tag{D.2}
$$

where the contour $C$ surrounds the negative real axis and the pole $x=-y$. The calculation can be performed by using the formula (B.22) after expanding the numerator of $w^{(2)}(y,x)$ with respect to the large $x$. In such a calculation, we assume that the unintegrated variable $y$ is outside the contour, and $-y$ is inside. By including the overall factors, it is rewritten as

$$s^{-1}\hat{\oint}\frac{d\sigma}{2\pi i}\sigma\hat{W}^{(2)}(\zeta,\sigma)=\hat{W}_1(\zeta)-a^{8/3}\frac{2^{1/3}s^{8/3}}{40}T^{4/3}+O(a^3) \tag{D.3}$$

where the integral symbol $\hat{\oint}\frac{d\sigma}{2\pi i}\sigma$ is used in the sense of

$$\hat{\oint}\frac{d\sigma}{2\pi i}\sigma=P_*^2 a\int_C\frac{dx}{2\pi i}.$$

Next, for $\hat{W}^{(4)}(\zeta_1,\sigma_1,\zeta_2,\sigma_2)$ and $\hat{W}_1(\zeta_1;\zeta_2,\sigma_2)$, we use the formulas:

$$\int_C\frac{dx_1}{2\pi i}\frac{1}{x_1-x_2}\frac{x_1^\alpha}{x_1+y_1} = -\frac{x_2^\alpha}{y_1+x_2} \quad (\alpha\notin\boldsymbol{Z}), \tag{D.4}$$

$$\int_C\frac{dx_1}{2\pi i}\frac{1}{x_1-x_2}\frac{x_1^n}{x_1+y_1} = -\frac{y_1^n}{y_1+x_2} \quad (n=0,1,2,\cdots), \tag{D.5}$$

which are derived from the formulas in the Appendix B, where we regard again that the unintegrated variables $x_2, y_1$ are outside the contour, and $-x_2, -y_1$ are inside. After some calculations, we have

$$\int_C\frac{dx_1}{2\pi i}w^{(4)}(y_1,x_1,y_2,x_2)=(w(y_1)+w(y_2))D_y(y_1,y_2)w^{(2)}(y,x_2), \tag{D.6}$$

$$s^{-1}\hat{\oint}\frac{d\sigma_1}{2\pi i}\sigma_1\hat{W}^{(4)}(\zeta_1,\sigma_1,\zeta_2,\sigma_2)=\hat{W}_1(\zeta_1;\zeta_2,\sigma_2)+O(a^{7/3}). \tag{D.7}$$

We can use this method also for a domain consisting of two flipped spins. For a preparation, we shall compute the amplitudes $\hat{W}_2(\zeta_1;\zeta_2,\sigma_2)$ and $\hat{W}_1(\zeta_1;\zeta_2,\sigma_2,\zeta_3,\sigma_3)$. From the analysis of the Schwinger-Dyson equations similar in the Appendix C, we obtain

$$\hat{W}_2(\zeta_1;\zeta_2,\sigma_2)=a^{10/3}\frac{s^{10/3}\sqrt{10c}}{320\cdot 2^{1/3}}w_2(y_1;y_2,x_2),$$

$$\begin{aligned} & w_2(y_1;y_2,x_2) \\ &\quad = (-w(y_1)^2-w(y_1)w(y_2)-w(y_2)^2+3T^{4/3})D_y(y_1,y_2)w^{(2)}(y,x_2), \end{aligned} \tag{D.8}$$

$$\begin{aligned} \hat{W}_1(\zeta_1;\zeta_2,\sigma_2,\zeta_3,\sigma_3) &= a^{4/3}\frac{5c^2s^{4/3}}{16\cdot 2^{1/3}}w_1(y_1;y_2,x_2,y_3,x_3), \\ w_1(y_1;y_2,x_2,y_3,x_3) &= (w(y_1)+w(y_2))D_y(y_1,y_2)w^{(4)}(y,x_2,y_3,x_3) \\ &\quad -D_y(y_2,y_3)w^{(2)}(y,x_2)D_y(y_3,y_1)w^{(2)}(y,x_3). \end{aligned} \tag{D.9}$$

Then it is easy to see that the following formulas hold:

$$\begin{aligned}\int_{C_1}\frac{dx_2}{2\pi i}\int_C\frac{dy_2}{2\pi i}w_1(y_1;y_2,x_2) &= \int_{C_1}\frac{dx_2}{2\pi i}\int_C\frac{dy_2}{2\pi i}w_1(y_2;y_1,x_2)\\ &= \int_{C_1}\frac{dx_2}{2\pi i}\int_C\frac{dx_1}{2\pi i}w_1(y_1,x_1;x_2)\\ &= \int_{C_1}\frac{dx_2}{2\pi i}\int_C\frac{dx_1}{2\pi i}w_1(y_1,x_2;x_1)\\ &= w_2(y_1)+2T^{4/3}w(y_1),\end{aligned}\tag{D.10}$$

where the contour $C_1$ wraps around the contour $C$. Moreover, after a straightforward but lengthy calculation, we can show that

$$\begin{aligned}&\int_{C_1}\frac{dx_2}{2\pi i}\int_C\frac{dy_2}{2\pi i}w_1(y_1;y_2,x_2,y_3,x_3)\\ &= \int_{C_1}\frac{dx_2}{2\pi i}\int_C\frac{dy_2}{2\pi i}w_1(y_2;y_3,x_3,y_1,x_2)\\ &= \int_{C_1}\frac{dx_2}{2\pi i}\int_C\frac{dx_1}{2\pi i}w_1(y_1,x_1;x_2,y_3,x_3)\\ &= \int_{C_1}\frac{dx_2}{2\pi i}\int_C\frac{dx_1}{2\pi i}w_1(y_1,x_2;x_1,y_3,x_3)\\ &= w_2(y_1;y_3,x_3)-2T^{4/3}D_y(y_1,y_3)w^{(2)}(y,x_3).\end{aligned}\tag{D.11}$$

Now by taking the overall factors into account, (D.10) and (D.11) are rewritten as

$$\begin{aligned}&s^{-1}\hat{\oint}\frac{d\sigma_2}{2\pi i}\sigma_2\left(\hat{\oint}\frac{d\zeta_2}{2\pi i}\hat{W}_1(\zeta_1;\zeta_2,\sigma_2)\right)\\ &= s^{-1}\hat{\oint}\frac{d\sigma_2}{2\pi i}\sigma_2\left(\hat{\oint}\frac{d\zeta_2}{2\pi i}\hat{W}_1(\zeta_2;\zeta_1,\sigma_2)\right)\\ &= s^{-1}\hat{\oint}\frac{d\sigma_2}{2\pi i}\sigma_2\left(\hat{\oint}\frac{d\sigma_1}{2\pi i}\hat{W}_1(\zeta_1,\sigma_1;\sigma_2)\right)\\ &= s^{-1}\hat{\oint}\frac{d\sigma_2}{2\pi i}\sigma_2\left(\hat{\oint}\frac{d\sigma_1}{2\pi i}\hat{W}_1(\zeta_1,\sigma_2;\sigma_1)\right)\\ &= \hat{W}_2(\zeta_1)+a^{8/3}\frac{2^{1/3}s^{8/3}}{40}T^{4/3}\hat{W}(\zeta_1)+O(a^{13/3}),\end{aligned}\tag{D.12}$$

$$\begin{aligned}&s^{-1}\hat{\oint}\frac{d\sigma_2}{2\pi i}\sigma_2\left(\hat{\oint}\frac{d\zeta_2}{2\pi i}\hat{W}_1(\zeta_1;\zeta_2,\sigma_2,\zeta_3,\sigma_3)\right)\\ &= s^{-1}\hat{\oint}\frac{d\sigma_2}{2\pi i}\sigma_2\left(\hat{\oint}\frac{d\zeta_2}{2\pi i}\hat{W}_1(\zeta_2;\zeta_3,\sigma_3,\zeta_1,\sigma_2)\right)\end{aligned}$$

$$
\begin{aligned}
&= s^{-1}\hat{\oint}\frac{d\sigma_2}{2\pi i}\sigma_2\left(\hat{\oint}\frac{d\sigma_1}{2\pi i}\hat{W}_1(\zeta_1,\sigma_1;\sigma_2,\zeta_3,\sigma_3)\right)\\
&= s^{-1}\hat{\oint}\frac{d\sigma_2}{2\pi i}\sigma_2\left(\hat{\oint}\frac{d\sigma_1}{2\pi i}\hat{W}_1(\zeta_1,\sigma_2;\sigma_1,\zeta_3,\sigma_3)\right)\\
&= \hat{W}_2(\zeta_1;\zeta_3,\sigma_3)-a^{8/3}\frac{2^{1/3}s^{8/3}}{40}T^{4/3}D_\zeta(\zeta_1,\zeta_3)\hat{W}^{(2)}(\zeta,\sigma_3)+O(a^{11/3}),
\end{aligned}
\tag{D.13}
$$

where the integral symbols are used in the sense of

$$
\hat{\oint}\frac{d\sigma_2}{2\pi i}\sigma_2=P_*^2a\int_{C_1}\frac{dx_2}{2\pi i},\quad \hat{\oint}\frac{d\zeta_2}{2\pi i}=P_*a\int_C\frac{dy_2}{2\pi i},\quad \hat{\oint}\frac{d\sigma_1}{2\pi i}=P_*a\int_C\frac{dx_1}{2\pi i}.
$$

These formulas show us that the domain consisting of two flipped spins is constructed by shrinking the domain between two microscopic domains consisting of a single flipped spin. Indeed, by substituting eq. (D.7), the first line of (D.12) becomes

$$
s^{-1}\hat{\oint}\frac{d\sigma_2}{2\pi i}\sigma_2\left(\hat{\oint}\frac{d\zeta_2}{2\pi i}\left(s^{-1}\hat{\oint}\frac{d\sigma_1}{2\pi i}\sigma_1\hat{W}^{(4)}(\zeta_1,\sigma_1,\zeta_2,\sigma_2)\right)\right).
$$

Thus the symbol $s^{-1}\hat{\oint}\frac{d\sigma_i}{2\pi i}\sigma_i$ is the single-spin flip operator, while $\hat{\oint}\frac{d\zeta_2}{2\pi i}$ shrinks the domain $\zeta_2$ to nothing in conformity with the original matrix-model operation.

# E Commutativity of the Mixing Matrix with Splitting and Merging Processes

Here, we present the calculations which leads to eq. (5.28)

$$
\left(\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)\vee\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)\right)_I=\left(\mathcal{M}\left(\frac{\delta}{\delta\hat{J}}\vee\frac{\delta}{\delta\hat{J}}\right)\right)_I
$$

for the first several components $I=A,B,1,2$, and eq. (5.29)

$$
\left(\wedge\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)\right)_{I,J}=\sum_{K,L}\mathcal{M}_{IK}\mathcal{M}_{JL}\left(\wedge\frac{\delta}{\delta\hat{J}}\right)_{K,L}
$$

for $(I,J)=(A,A),(B,B),(A,1),(B,1),(A,2),(B,2),(1,1)$.

First, we consider about eq. (5.28). For $I=A,B$, it is trivial from the definition of $\mathcal{M}$ and $\vee$

$$
\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_A = \frac{\delta}{\delta\hat{J}_A(\zeta)},
$$

$$
\begin{aligned}
\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_B &= \frac{\delta}{\delta\hat{J}_B(\sigma)}, \\
\left(\frac{\delta}{\delta\hat{J}}\vee\frac{\delta}{\delta\hat{J}}\right)_A &= -\partial_\zeta\left(\frac{\delta^2}{\delta\hat{J}_A(\zeta)^2}\right), \qquad \text{(E.1)} \\
\left(\frac{\delta}{\delta\hat{J}}\vee\frac{\delta}{\delta\hat{J}}\right)_B &= -\partial_\sigma\left(\frac{\delta^2}{\delta\hat{J}_B(\sigma)^2}\right). \qquad \text{(E.2)}
\end{aligned}
$$

For $I=1$, using

$$
\begin{aligned}
\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_1 &= \sqrt{10c}\left(\frac{\delta}{\delta\hat{J}_A(\zeta)}+\frac{\delta}{\delta\hat{J}_B(\sigma)}\right)+\frac{\delta}{\delta\hat{J}_1(\zeta,\sigma)}, \qquad \text{(E.3)} \\
\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)_1 &= -2\left(\frac{\delta}{\delta J_A(\zeta)}\partial_\zeta\frac{\delta}{\delta J_1(\zeta,\sigma)}+\frac{\delta}{\delta J_B(\sigma)}\partial_\sigma\frac{\delta}{\delta J_1(\zeta,\sigma)}\right), \qquad \text{(E.4)}
\end{aligned}
$$

we have

$$
\begin{aligned}
&\left(\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)\vee\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)\right)_1 \\
&= -2\left(\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_A\partial_\zeta\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_1+\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_B\partial_\sigma\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_1\right) \\
&= -2\sqrt{10c}\left(\frac{\delta}{\delta\hat{J}_A(\zeta)}\partial_\zeta\frac{\delta}{\delta\hat{J}_A(\zeta)}+\frac{\delta}{\delta\hat{J}_B(\sigma)}\partial_\sigma\frac{\delta}{\delta\hat{J}_B(\sigma)}\right) \\
&\quad -2\left(\frac{\delta}{\delta\hat{J}_A(\zeta)}\partial_\zeta\frac{\delta}{\delta\hat{J}_1(\zeta,\sigma)}+\frac{\delta}{\delta\hat{J}_B(\sigma)}\partial_\sigma\frac{\delta}{\delta\hat{J}_1(\zeta,\sigma)}\right). \qquad \text{(E.5)}
\end{aligned}
$$

On the other hand,

$$
\left(\mathcal{M}\left(\frac{\delta}{\delta\hat{J}}\vee\frac{\delta}{\delta\hat{J}}\right)\right)_1=\sqrt{10c}\left(\left(\frac{\delta}{\delta\hat{J}}\vee\frac{\delta}{\delta\hat{J}}\right)_A+\left(\frac{\delta}{\delta\hat{J}}\vee\frac{\delta}{\delta\hat{J}}\right)_B\right)+\left(\frac{\delta}{\delta\hat{J}}\vee\frac{\delta}{\delta\hat{J}}\right)_1,
$$

which is nothing but the r.h.s. of (E.5). Similarly, we can show the validity of the formula for $I=2$ by noticing the following identities

$$
\begin{aligned}
&\sum_{j=1}^{2}\frac{\delta}{\delta\hat{J}_A(\zeta_j)}\partial_{\zeta_j}D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta\hat{J}_1(\zeta,\sigma_1)} \\
&\quad = D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta\hat{J}_A(\zeta)}\partial_\zeta\frac{\delta}{\delta\hat{J}_1(\zeta,\sigma_1)}-D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta J_A(\zeta)}D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta J_1(\zeta,\sigma_1)},
\end{aligned}
$$

$$
\sum_{j=1}^{2}\frac{\delta}{\delta\hat{J}_A(\zeta_j)}\partial_{\zeta_j}D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta\hat{J}_A(\zeta)}+\left(D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta\hat{J}_A(\zeta)}\right)^2=D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta\hat{J}_A(\zeta)}\partial_\zeta\frac{\delta}{\delta\hat{J}_A(\zeta)}. \qquad \text{(E.6)}
$$

Next, we consider eq. (5.29). Note that $\mathcal{M}$ takes the upper-triangular form: $\mathcal{M}_{IJ} = 0$ for $I < J$. For $(I, J) = (A, A)$,

$$\begin{aligned}\left(\wedge\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)\right)_{A,A}(\zeta;\zeta') &= \left(\wedge\frac{\delta}{\delta\hat{J}}\right)_{A,A}(\zeta;\zeta') \\ &= \sum_{K,L}\mathcal{M}_{AK}\mathcal{M}_{AL}\left(\wedge\frac{\delta}{\delta\hat{J}}\right)_{K,L}(\zeta;\zeta'),\end{aligned} \tag{E.7}$$

because $\mathcal{M}_{AL} = \delta_{A,L}$. For $(I, J) = (A, 1)$,

$$\begin{aligned}&\left(\wedge\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)\right)_{A,1}(\zeta';\zeta_1,\sigma_1) = -\partial_{\zeta'}\partial_{\zeta_1}D_z(\zeta_1,\zeta')\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_1(z,\sigma_1) \\ &= -\sqrt{10c}\,\partial_{\zeta'}\partial_{\zeta_1}D_z(\zeta_1,\zeta')\frac{\delta}{\delta\hat{J}_A(z)} - \partial_{\zeta'}\partial_{\zeta_1}D_z(\zeta_1,\zeta')\frac{\delta}{\delta\hat{J}_1(z,\sigma_1)}.\end{aligned} \tag{E.8}$$

On the other hand,

$$\begin{aligned}&\sum_{K,L}\mathcal{M}_{AK}\mathcal{M}_{1L}\left(\wedge\frac{\delta}{\delta\hat{J}}\right)_{K,L}(\zeta';\zeta_1,\sigma_1) \\ &= \sqrt{10c}\left(\wedge\frac{\delta}{\delta\hat{J}}\right)_{A,A}(\zeta';\zeta_1) + \left(\wedge\frac{\delta}{\delta\hat{J}}\right)_{A,1}(\zeta';\zeta_1,\sigma_1),\end{aligned} \tag{E.9}$$

which is nothing but the r.h.s. of (E.8).

Similarly, for $(I, J) = (A, 2), (1, 1)$, checking the formula is straightforward by utilizing the identities such as

$$\begin{aligned}&D_z(\zeta_1,\zeta_2)\partial_z D_w(\zeta',z)\frac{\delta}{\delta\hat{J}_A(w)} \\ &= \partial_{\zeta_1}D_z(\zeta',\zeta_1)D_w(z,\zeta_2)\frac{\delta}{\delta\hat{J}_A(w)} + \partial_{\zeta_2}D_z(\zeta',\zeta_2)D_w(z,\zeta_1)\frac{\delta}{\delta\hat{J}_A(w)},\end{aligned} \tag{E.10}$$

$$D_z(\zeta_1,\zeta_1')D_w(\zeta_1,\zeta_1')D_\zeta(z,w)\frac{\delta}{\delta\hat{J}_A(\zeta)} = \partial_{\zeta_1}\partial_{\zeta_1'}D_z(\zeta_1,\zeta_1')\frac{\delta}{\delta\hat{J}_A(z)}. \tag{E.11}$$

The validity for $(I, J) = (B, 1), (B, 2)$ is obvious from the symmetry of $\mathcal{M}$ with respect to $A \leftrightarrow B$.

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